Math 5B Summer 2006 Midterm

Name: Answer Key

Perm: ______

Your work must be neat and complete in order to receive full credit.

You must label all axes on all graphs for full credit.

All answers should be left in exact form; only exact decimals are fine. For instance, 1/10 and 0.1 are both acceptable, but 3.14 is not if the answer is π.

If you need more space for work, use the back of the paper, but indicate on the front that there is work on the back.

(1) (15 points) Let $\mathbf{u} = (2, -1, 2)$ and $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$

(a) (3 points) Find $\mathbf{u} + 2\mathbf{v}$.

$(-2, 5, 14)$

(b) (4 points) Find $\mathbf{u} \times \mathbf{v}$.

$$\left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & -1 & 2 & 2 & -1 \\ -2 & 3 & 6 & -2 & 3 \end{array}\right. = -6\vec{i} - 4\vec{j} + 6\vec{k} - (12\vec{j} + 6\vec{i} + 2\vec{k})$$

$$= \boxed{-12\vec{i} - 16\vec{j} + 4\vec{k}}$$

(c) (4 points) Find $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$.

$\boxed{0}$ since by definition $\vec{u} \times \vec{v} \perp \vec{u}$

(d) (4 points) Find $\angle(\mathbf{u}, \mathbf{v})$. You may leave your solution as an Arcsin or Arccos.

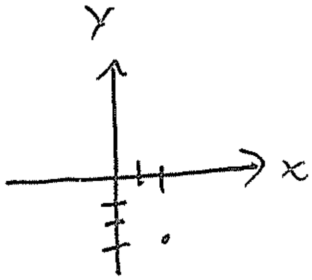
$|\vec{u}| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$ $\quad$ $|\vec{v}| = \sqrt{(-2)^2 + 3^2 + 6^2} = 7$

$\vec{u} \cdot \vec{v} = 2(-2) - 1(3) + 2(6) = 5$

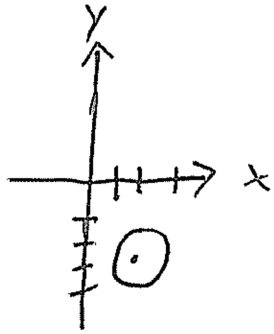
$5 = 3 \cdot 7 \cdot \cos(\theta)$ $\quad$ $\theta = \angle(\vec{u}, \vec{v}) = \boxed{\mathrm{Arccos}\left(\frac{5}{21}\right)}$

(2) (15 points) (a) (10 points) Sketch at least 5 level curves of $z^2 = (x-2)^2 + (y+3)^2$. You must label your axes.

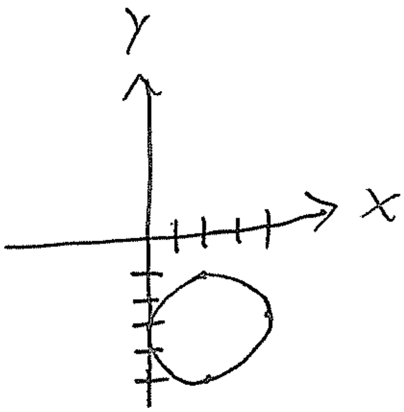
$Z = 0$



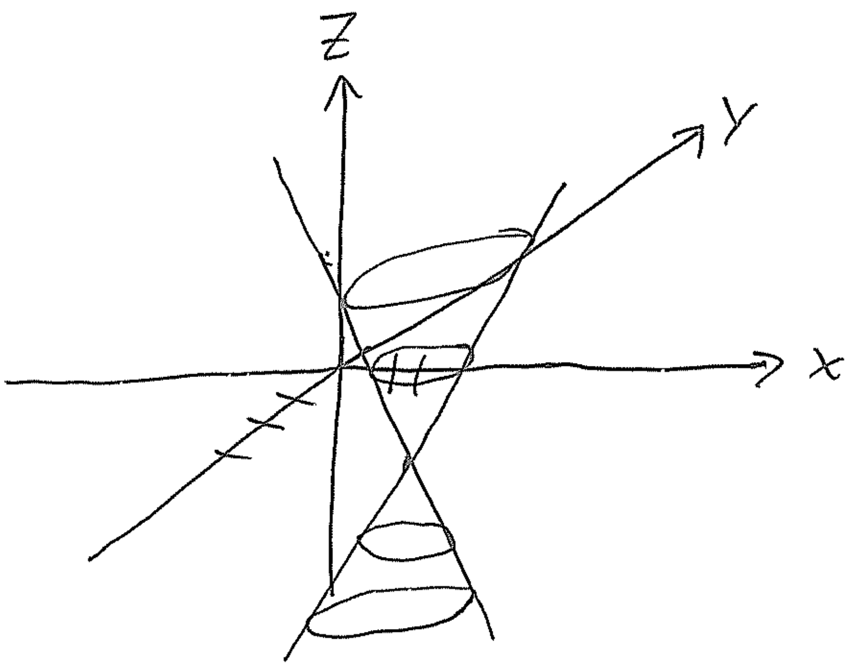
$Z = \pm 1$



$Z = \pm 2$



(b) (5 points) Sketch the surface. You must label your axes.



(3) (15 points) Find any tangents to the curve $x(t) = 2 + t$, $y(t) = t^2$, $z(t) = te^t$ which pass through $(5, 5, 5e)$.

$x'(t_0) = 1$ $y'(t) = 2t$ $z'(t) = te^t + e^t$

$x(t_0) = 2 + t_0$ $y(t_0) = t_0^2$ $z(t_0) = t_0 e^{t_0}$

$x'(t_0) = 1$ $y'(t_0) = 2t_0$ $z'(t_0) = t_0 e^{t_0} + e^{t_0}$

At $t = t_0$, the tangent line is:

$x(t) = 2 + t_0 + t$ $y(t) = t_0^2 + 2t_0 t$ $z(t) = t_0 e^{t_0} + (t_0 e^{t_0} + e^{t_0})t$

so $5 = 2 + t_0 + t$ $5 = t_0^2 + 2t_0 t$ $5e = t_0 e^{t_0} + (t_0 e^{t_0} + e^{t_0})t$

$\Downarrow$ $\Downarrow$

$t = 3 - t_0 \longrightarrow 5 = t_0^2 + 2t_0(3 - t_0)$

so $5 = t_0^2 + 6t_0 - 2t_0^2 \Rightarrow t_0^2 - 6t_0 + 5 = 0$

$(t_0 - 5)(t_0 - 1) = 0$

$t_0 = 5$ or $t_0 = 1$

~~check last equation: if $t_0 = 5$ $5e = 5e^5 + ($~~

if $t_0 = 5$ $t = -2$ check last equation:

if $t_0 = 1$ $t = 2$ $5e \stackrel{?}{=} 5e^5 + (5e^5 + e^5)(-2)$ these are not equal so $t_0 = 5$ does not work.

$5e \stackrel{?}{=} 1e^1 + (1e^1 + e^1)(2)$ ✓ so the tangent line is: $\boxed{x(t) = 3 + t \quad y(t) = 1 + 2t}$ $\boxed{z(t) = e + 2et}$

(4) (30 points) Let $x(u,v) = 2uv$, $w(u,v) = u^2v^2$, $y(u,v) = u^2 - v^2$, and $z(u,v) = u + v$.
(a) (10 points) Find the closest linear approximation to the inverse function of $x(u,v)$ and $y(u,v)$ at $(u,v) = (2,1)$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} 2v & 2u \\ 2u & -2v \end{bmatrix} \qquad \frac{\partial(x,y)}{\partial(u,v)}\bigg|_{(2,1)} = \begin{bmatrix} 2 & 4 \\ 4 & -2 \end{bmatrix}$$

So $$\frac{\partial(u,v)}{\partial(x,y)}\bigg|_{(u,v)=(2,1)} = \left(\begin{bmatrix} 2 & 4 \\ 4 & -2 \end{bmatrix}\right)^{-1} = \frac{1}{2(-2)-4(4)}\begin{bmatrix} -2 & -4 \\ -4 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{10} \end{bmatrix}}$$

(b) (5 points) Find $\left(\frac{\partial u}{\partial x}\right)_y\Big|_{(x,y)=(4,3)}$. (Hint: Don't take determinants of 4x4 matrices)

$\boxed{\frac{1}{10}}$ from part (a) since

$x(2,1) = 4$ and $y(2,1) = 3$

(c) (10 points) Find $\frac{\partial(w,z)}{\partial(x,y)}\Big|_{(4,3)}$. I mean a matrix here, NOT its determinant. (Hint: Use part a).

By the chain rule and

$$\frac{\partial(w,z)}{\partial(x,y)}\bigg|_{(4,3)} = \frac{\partial(w,z)}{\partial(u,v)}\bigg|_{(2,1)} \frac{\partial(u,v)}{\partial(x,y)}\bigg|_{(4,3)}$$

$$\frac{\partial(w,z)}{\partial(u,v)} = \begin{bmatrix} 2uv^2 & 2u^2v \\ 1 & 1 \end{bmatrix}$$

(d) (5 points) Approximate w when $x = 3.9$ and $y = 3.1$. (Hint: Use part c).

So $$\frac{\partial(w,z)}{\partial(x,y)}\bigg|_{(4,3)} = \begin{bmatrix} 4 & 8 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{10} \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 0 \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}}$$

d) $$w \approx w\Big|_{(x,y)=(4,3)} + 2(x-4) + 0(y-3)$$

$$= w(2,1) + 2(-.1) + 0(.1) = 4 - 0.2 = \boxed{3.8}$$

(5) (25 points) Let $f(x,y) = \begin{cases} \dfrac{e^{x^2+y^2}-1}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$

(a) (10 points) Where is f(x,y) continuous?

It is continuous at least at all $(x,y) \neq (0,0)$

Check at $(0,0)$:

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{e^{x^2+y^2}-1}{x^2+y^2} = \lim_{r\to 0^+} \frac{e^{r^2}-1}{r^2} \overset{\text{by L'Hopital's}}{=} \lim_{r\to 0^+} \frac{\cancel{2r}e^{r^2}}{\cancel{2r}} = 1 = f(0,0).$$

So $\boxed{f \text{ is continuous everywhere}}$.

(b) (5 points) Find the total differential of f(x,y) where it exists.

$$df = \frac{(x^2+y^2)\left[e^{x^2+y^2}(2x\,dx+2y\,dy)\right] - (e^{x^2+y^2}-1)(2x\,dx+2y\,dy)}{(x^2+y^2)^2}$$

$$= \boxed{\frac{2x\left[(x^2+y^2-1)e^{x^2+y^2}+1\right]}{(x^2+y^2)^2}\,dx + \frac{2y\left[(x^2+y^2-1)e^{x^2+y^2}+1\right]}{(x^2+y^2)^2}\,dy}$$

(c) (10 points) Where does the total differential exist?

From (b) each partial is clearly continuous at all $(x,y) \neq (0,0)$. Check at $(0,0)$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{2x\left[(x^2+y^2-1)e^{x^2+y^2}+1\right]}{(x^2+y^2)^2} = \lim_{r\to 0^+} \frac{2\cancel{r}\cos(\theta)\left[(r^2-1)e^{r^2}+1\right]}{r^{\cancel{4}3}}$$

$$= \lim_{r\to 0^+} \frac{2\cos(\theta)\left[r^2e^{r^2}-e^{r^2}+1\right]}{r^3} \overset{\text{L'Hopital's}}{=} \lim_{r\to 0^+} 2\cos(\theta)\frac{2r^3e^{r^2}+\cancel{2re^{r^2}}-\cancel{2re^{r^2}}}{3r^2}$$

$$= \lim_{r\to 0^+} 2\cos(\theta)\cdot\frac{2}{3}re^{r^2} = 0.$$

Similarly $\lim_{\substack{x\to 0\\ y\to 0}} \frac{\partial f}{\partial y} = \lim_{r\to 0^+} 2\sin(\theta)\cdot\frac{2}{3}re^{r^2} = 0$

$\boxed{\text{So the differential exists everywhere.}}$

Extra Credit: Find the inverse function of $x = u^2 + v^2, y = 2uv$. What is the domain of the inverse function?

for $x+y \geq 0$

$x+y = u^2 + 2uv + v^2 = (u+v)^2$

$x-y = u^2 - 2uv + v^2 = (u-v)^2$

$u+v = \sqrt{x+y}$

$u-v = \sqrt{x-y}$

for $x-y \geq 0$

$2u = \sqrt{x+y} + \sqrt{x-y}$

$2v = \sqrt{x+y} - \sqrt{x-y}$

$$\boxed{u = \frac{\sqrt{x+y} + \sqrt{x-y}}{2} \qquad v = \frac{\sqrt{x+y} - \sqrt{x-y}}{2}}$$

~~It is defined~~ Its domain is

$\{(x,y) \in \mathbb{R}^2 : x+y \geq 0 \text{ and } x-y \geq 0\}$

$= \boxed{\{(x,y) \in \mathbb{R}^2 ; x \geq 0 \text{ and } |y| < x\}}$