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Absolute Convergence:

What if we have $\sum_{n=1}^{\infty} a_n$ where some a_n's are positive & some are negative, but not in an alternating fashion.

eg $\sum_{n=0}^{\infty} \cos(n) = 1 + \cos(1) + \cos(2) + \cos(3) + \cos(4) + \cos(5) + \cdots$

$\quad 1 > 0,\ \cos(1) > 0,\ \cos(2) < 0,\ \cos(3) < 0,\ \cos(4) < 0,\ \cos(5) > 0$

Diverges by n^{th} Term test since $\lim_{n\to\infty} \cos(n) = \text{D.N.E} \neq 0$.

eg $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ $\left(\sim \sum_{n=1}^{\infty} \frac{1}{n^2}\right)$

Should converge b/c $\cos(n) \leq 1$ for every n, but we cannot simply use the comparison test since some terms are negative.

Def $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |a_n|$ converges.

Thm If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (or <u>converges absolutely</u>), then it converges.

eg $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ converges absolutely, & thus converges.

pf $\sum_{n=1}^{\infty} \left|\frac{\cos(n)}{n^2}\right|$ converges by comp. test w/ $\sum_{n=1}^{\infty} \frac{1}{n^2}$

since $0 \leq \left|\frac{\cos(n)}{n^2}\right| \leq \frac{1}{n^2}$ & $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since it is a p-series w/ $p = 2 > 1$.

Extra Credit: Does $\sum_{n=1}^{\infty} \frac{\cos(n)}{n}$ Converge or Diverge? Justify.

Def If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges, $\sum_{n=1}^{\infty} a_n$ is <u>conditionally convergent</u>.

eg $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent: it converges by the Alternating Series test, But $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges. (b/c it's Harmonic)

Ratio Test Suppose $a_n \neq 0$ for all n, and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. Then

a) $L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is absolutely convergent

b) $L = 1 \Rightarrow$ Inconclusive (we know nothing about $\sum_{n=1}^{\infty} a_n$)

c) $L > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ Diverges.

eg $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ = "n factorial"
$0! = 1$, $1! = 1$, $2! = 2\cdot 1 = 2$, $3! = 3\cdot 2\cdot 1 = 6$, $4! = 4\cdot 3\cdot 2\cdot 1 = 24$

eg $\sum_{n=1}^{\infty} \frac{2^n}{n!}$: Ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}/(n+1)!}{2^n/n!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{2}{n+1}\right| \quad \text{since } (n+1)! = (n+1)\cdot n!$$

$$= 0 < 1.$$

$\Rightarrow$ Converges (Absolutely)

eg $\sum_{n=1}^{\infty} \frac{n^n}{n!}$: Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}/(n+1)!}{n^n/n!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)\left(\frac{n+1}{n}\right)^n \cdot n!}{(n+1)!}\right|$$

$$= \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)^n\right| = e > 1$$

$\therefore$ Diverges.

eg product of 1^{st} n positive odd integers $= 1\cdot3\cdot5\cdots(2n-1)$
product of 1^{st} n positive even integers $= 2\cdot4\cdot6\cdots(2n)$

Formulas: $2\cdot4\cdot6\cdots2n = (2\cdot1)(2\cdot2)(2\cdot3)\cdots(2\cdot n)$

$$= 2^n(1\cdot2\cdot3\cdots n)$$

$$= 2^n\cdot n!$$

$$1\cdot3\cdot5\cdots(2n-1) = \frac{1\cdot2\cdot3\cdot4\cdots2n}{2\cdot4\cdot6\cdots2n} = \frac{(2n)!}{2^n\, n!}$$

eg $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{1\cdot3\cdot5\cdots(2n-1)} = \sum_{n=1}^{\infty} \frac{n^2\cdot2^n\cdot n!}{(2n)!}$ Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2/1\cdot3\cdot5\cdots(2n-1)(2n+1)}{n^2/1\cdot3\cdot5\cdots(2n-1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\left(\frac{n+1}{n}\right)^2}{2n+1}\right| = \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)^2}{2n+1}\right| = 0 < 1$$

$\therefore$ Converges (Absolutely)

Root Test Given $\displaystyle\sum_{n=1}^{\infty} a_n$ s.t. $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = R$. Then

$R<1 \Rightarrow \displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely.

$R>1 \Rightarrow \displaystyle\sum_{n=1}^{\infty} a_n$ Diverges.

$R=1 \Rightarrow$ Inconclusive.

eg $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(\log n)^n}$ Root test (since we see an n^{th} power of a function of n)

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{\ln n}\right)^n} = \lim_{n\to\infty}\left|\frac{1}{\ln n}\right| = 0 < 1.$$

$\therefore$ Converges.

Operations on Series.

Let $\{a_n\}_{n\geq0}$ be a sequence & $\{b_n\}_{n\geq0}$ be another
Let $c\neq0$ be a real number. Then

1) $\displaystyle\sum_{n=0}^{\infty} ca_n = c\sum_{n=0}^{\infty} a_n$ converges $\Longleftrightarrow$ $\displaystyle\sum_{n=0}^{\infty} a_n$ converges.

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2) If $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ both converge, then
$\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ converges.

3) If $\sum_{n=1}^{\infty} a_n$ converges & $\sum_{n=1}^{\infty} b_n$ diverges, then
$\sum_{n=1}^{\infty}(a_n+b_n)$ Diverges.

4) If $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ both diverge, then
$\sum_{n=1}^{\infty}(a_n+b_n)$ may converge or diverge.

eg $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ a_n looks like $\frac{\sqrt{n}}{n} = n^{-1/2}$ & $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

In fact, $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n} - \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$

both diverge $\frac{\sqrt{n+1}}{n} > \frac{1}{\sqrt{n}}$ & $\sum \frac{1}{\sqrt{n}}$
diverges by p-series: $p = 1/2 \le 1$.

But a sum/difference of 2 divergent sequences may still converge:

$$a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{n(\sqrt{n+1}+\sqrt{n})}$$

$$= \frac{n+1-n}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \approx \frac{1}{n^{3/2}}$$

$$0 \le a_n = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \le \frac{1}{2n\sqrt{n}} = \frac{1}{2}\cdot\frac{1}{n^{3/2}}.$$

and $\sum_{n=1}^{\infty} \frac{1}{2}\frac{1}{n^{3/2}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges by p-test:
$p = 3/2 > 1$.

$\therefore$ by Comparison test, $\sum_{n=1}^{\infty} a_n$ converges.