(45)

eg $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad c_n = 1 = \frac{f^n(0)}{n!}$ for all n.

$\Rightarrow f^{(n)}(0) = n!\, c_n = n!$

$\therefore$ The 32^{nd} Derivative of $\frac{1}{1-x}$, evaluated at $x=0$, is $32!$

OTHER METHODS FOR COMPUTING TAYLOR SERIES

Thm If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R, then

$f'(x) = \sum_{n=0}^{\infty} c_n n(x-a)^{n-1} = \sum_{n=0}^{\infty} (n+1)c_{n+1}(x-a)^n$ has R of C $= R$, and

$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C = C + \sum_{n=1}^{\infty} \frac{c_{n-1}}{n}(x-a)^n$ has R of C $= R$.

But convergence at endpoints of I.C. may change when we integrate / Differentiate.

eg Find the Taylor Series for $f(x) = \tan^{-1}x$.

$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = 1 - x^2 + x^4 - x^6 + x^8 - \ldots$

for $|-x^2| < 1$, thus for $|x| < 1$. ($R = 1$, I.C. $= (-1,1)$)

$f(x) = \int \frac{1}{1+x^2}dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C$

$C = f(0) = \tan^{-1}(0) = 0$. $\therefore \boxed{\tan^{-1}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}$

To find I.C., R is still 1, test endpoints.

$x = -1 \quad \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{2n+1}}{2n+1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges by A.S.T.

$x = 1 \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges by A.S.T. $\Rightarrow \boxed{\text{I.C.} = [-1,1]}$

$\therefore x = 1: \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \tan^{-1}(1) = \boxed{\pi/4}$

eg $f(x) = \sin x$

$$
\left.\begin{array}{ll}
f^0(x) = \sin x & f^0(0) = 0 \\
f^1(x) = \cos x & f^1(0) = 1 \\
f^2(x) = -\sin x & f^2(1) = 0 \\
f^3(x) = -\cos x & f^3(0) = -1 \\
f^4(x) = \sin x & f^4(0) = 0 \\
\vdots &
\end{array}\right\}
\boxed{\begin{aligned}
\sin x &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} \cdots \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\end{aligned}}
$$

To find I.C. / R.C. Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}/(2n+3)!}{x^{2n+1}/(2n+1)!} \right| = |x^2| \lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+3)} \right| = 0$$

$0 < 1$ for all x $\Rightarrow \boxed{R = \infty,\ I.C. = (-\infty, \infty)}$

eg $f(x) = \cos x = \frac{d}{dx}(\sin x)$

$$\cos x = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R.C. = \infty,\ I.C. = (-\infty, \infty)}$$

eg $f(x) = \sqrt{1+x}$ $\qquad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\qquad f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2}$

$f'''(x) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\cdot\left(\frac{-3}{2}\right)(1+x)^{-5/2}$

$\vdots$

$$f^n(x) = \underbrace{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)\cdots\left(\frac{-(2n-3)}{2}\right)}(1+x)^{-\frac{(2n-1)}{2}} = \frac{(-1)^{n-1}\, 1\cdot 3\cdot 5 \cdots (2n-3)}{2^n}(1+x)^{-\frac{(2n-1)}{2}}$$

= n terms

n-1 ~1 factors

$$\Rightarrow f^n(0) = \frac{(-1)^{n-1}\, 1\cdot 3\cdot 5 \cdots (2n-3)}{2^n} = \frac{(-1)^{n-1}(2n-2)!}{2^n \cdot 2^{n-1}(n-1)!}$$

=

$$= \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}(n-1)!} \qquad (n \geq 2)$$

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$$\therefore \boxed{\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}1\cdot3\cdot5\cdots(2n-3)}{2^n\, n!}x^n = 1 + \frac{x}{2} + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}(n-1)!\,n!}x^n}$$

Ratio Test to find R.C. & I.C.

$$\lim_{n\to\infty}\left|\frac{x^{n+1}1\cdot3\cdots(2n-1)/2^{n+1}(n+1)!}{x^n 1\cdot3\cdots(2n-3)/2^n\, n!}\right| = \lim_{n\to\infty}\left|\frac{2n-1}{2(n+1)}\right||x| = |x| < 1$$

so $\boxed{R=1}$

convergence at endpoints: $x=-1$, $x=1$.

$$x=-1:\ -\sum_{n=2}^{\infty}\frac{1\cdot3\cdot5\cdots(2n-3)}{2^n\, n!} = -\sum_{n=2}^{\infty}\frac{1\cdot3\cdot5\cdots(2n-3)}{2\cdot4\cdot6\cdots(2n-2)(2n)}$$

Diverges (?) - Extra Credit.

$$x=1:\ \sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot3\cdots(2n-3)}{2\cdot4\cdot6\cdots(2n)} \quad \text{converges by A.S.T.}$$

$$b_n = \frac{1\cdot3\cdots(2n-3)}{2\cdot4\cdot6\cdots(2n)} > \frac{1\cdot3\cdots(2n-3)(2n-1)}{2\cdot4\cdots(2n)(2n+2)} = b_{n+1}$$

$$\text{and } \lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1\cdot3\cdots(2n-3)}{2\cdot4\cdots(2n)} \le \lim_{n\to\infty}\frac{1}{2n} = 0.$$

and $\lim_{n\to\infty} b_n \ge 0$ so it must be 0 by squeeze Theorem

$\boxed{I.C. = (-1,1]}$

Application $\sqrt{2} = f(1) = 1 + \frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot3\cdot5\cdots(2n-3)}{2^n\, n!}$

$$= 1 + \frac{1}{2} - \frac{1}{2\cdot4} + \frac{1\cdot3}{2\cdot4\cdot6} - \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8} + \cdots$$

Thm (Taylor's Formula with Remainder) (6.17)

$$f(x) = \sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n$$

ie, $R_N = f(x) - \sum_{n=0}^{N}\frac{f^n(a)}{n!}(x-a)^n$ = ERROR / REMAINDER from N^{th} partial sum of Taylor Series for $f(x)$.

If the 1^{st} $n+1$ derivatives of $f(x)$ are continuous on $(a-r_0, a+r_0)$ then there is an x_1 in $(a-x, a+x)$ s.t. $R_N(x) = \frac{f^{n+1}(x_1)}{(n+1)!}(x-a)^{N+1}$ (x_1 depends on x).

If $|f^{(N+1)}(x)| \leq M$ for x in $(a-r_0, a+r_0)$ then
$|R_N(x)| \leq \frac{M}{(n+1)!}(x-a)^{N+1}$ for all x in $(a-r_0, a+r_0)$.

$\therefore$ To show the Taylor series for $f(x)$ converges to $f(x)$, we must show $\lim_{N\to\infty} |R_N(x)| = 0$ (when x is in I.C.)

eg $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad R_N(x) = e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}\right)$

By Taylor's Formula, $R_N(x) = \frac{f^{N+1}(x_1)}{(N+1)!}(x-0)^{N+1}$ for some x_1 w/ $0 < x_1 < x$

$$= \frac{e^{x_1}}{(N+1)!} x^{N+1}$$

So $x_1 < x \Rightarrow |R_N(x)| \leq \frac{e^x x^{N+1}}{(N+1)!}$ for all N.

Thus $\lim_{N\to\infty} |R_N(x)| \leq \underbrace{\lim_{N\to\infty} \frac{e^x x^{N+1}}{(N+1)!} = 0}_{\text{b/c The Series } \sum_{N=0}^{\infty} \frac{e^x x^{N+1}}{(N+1)!} \text{ converges by Ratio Test.}}$ (x fixed)

$\Rightarrow \lim_{N\to\infty} |R_N(x)| = 0$ & thus for any x, the error in approximating e^x by $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}$ approaches 0 as $N \to \infty$.