(For Taylor's Formula with Remainder, see notes from previous Lecture (p.47-48))

(49) <u>Sequences and Series of functions</u>.

$$\sum_{n=0}^{\infty} u_n(x) = u_0(x) + u_1(x) + u_2(x) + \cdots$$

is an infinite series of <u>functions of x</u>, $u_n(x)$

eg Power Series $\sum_{n=0}^{\infty} c_n x^n$, $u_n(x) = c_n x^n$.

For each value of x, substituting that value for x yields an infinite series of numbers, which may or may not converge.
If it converges, the value of the sum depends on the value of x we plugged in.
ie, the series $\sum_{n=0}^{\infty} u_n(x)$ is again a function of x.

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \quad \text{(when it converges)}$$

eg $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (-1<x<1)$

when $x = -1/2$, $\frac{1}{1-(-1/2)} = \sum_{n=0}^{\infty} (-\frac{1}{2})^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

$\quad = 2/3$.

eg $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. If we plug in $x = 3$, the series $\sum_{n=0}^{\infty} \frac{3^n}{n!}$ converges to e^3.

eg $\sum_{n=0}^{\infty} \cos(nx)$, $\sum_{n=0}^{\infty} (\ln x)^n$, $\sum_{n=0}^{\infty} n^x$

are series of functions (but not power series)

- Convergence of infinite series of functions is defined as for infinite series of numbers.
  Let $S_N(x) = \sum_{n=0}^{N} u_n(x)$ be the N^{th} partial sum.

  Then $\sum_{n=0}^{\infty} u_n(x) = \lim_{N\to\infty} S_N(x)$ (if the limit exists)

  or the sum Diverges (if the limit $= \pm\infty$ or D.N.E)

UNIFORM CONVERGENCE.

When we have a sequence of functions converging to a single function (eg. $S_N(x) \to U(x)$), this means that, for each x, $\lim_{N\to\infty} S_N(x) \to U(x)$:

for $x = a$,

$\{S_N(a)\}_{N\geq 0}$ = sequence of numbers converging to $U(a)$

In general, the rate at which this sequence converges may depend on x.

eg $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$, for $-1 < x < 1$, but it converges at a slower rate when x is closer to 1.

<u>Def</u> Let $\{f_n(x)\}_{n\geq 0}$ be a sequence of functions. This sequence <u>converges uniformly</u> to a function $f(x)$ on the Domain E if

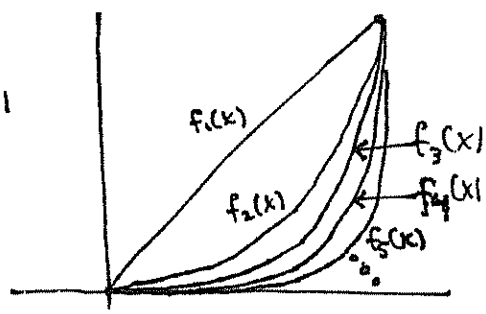
for any $\varepsilon > 0$, There is an N s.t.

$n \geq N \Rightarrow |f(x) - f_n(x)| < \varepsilon$ for all x in E.

The series $\sum_{n=0}^{\infty} U_n(x)$ <u>converges uniformly</u> to $U(x)$ if the sequence of partial sums $\{S_n(x)\}_{n\geq 0}$ converges uniformly to $U(x)$.

eg Let $f_n(x) = x^n$ for $0 \leq x \leq 1$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$



Here, the $\{f_n(x)\}$ <u>do not</u> converge uniformly to the piecewise function

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1. \end{cases}$$

Since if $\varepsilon = 1/2$, for any N, there is still an $x < 1$ with $x^N > 1/2$.

<u>Alternative Definition</u> $\{f_n(x)\}_{n\geq 0}$ converges uniformly to $f(x)$ on the Domain E $\iff$
$$\lim_{n\to\infty} \max_E(|f_n(x)-f(x)|)=0$$

where $\max_E(|f_n(x)-f(x)|)$ = "max. value" (least upper bound) of $|f_n(x)-f(x)|$ when x is in E.

eg $\sum_{n=0}^{\infty} x^n$ converges to $f(x)=\frac{1}{1-x}$ uniformly on $[-1/2, 1/2]$ (but not uniformly on $(-1,1)$)

<u>proof</u> We must show that the maximum possible value of $|f(x)-S_N(x)|$ tends towards 0 as $N\to\infty$.

$$|f(x)-S_N(x)| = \left|\frac{1}{1-x}-\frac{1-x^{N+1}}{1-x}\right| = \left|\frac{x^{N+1}}{1-x}\right|$$

$$\left.\begin{array}{l}|x|\leq 1/2 \Rightarrow |x^{N+1}|\leq \left(\frac{1}{2}\right)^{N+1} \\ x\leq 1/2 \Rightarrow 1-x\geq 1-1/2=1/2\end{array}\right\} \quad \frac{|x^{N+1}|}{|1-x|}\leq \frac{(1/2)^{N+1}}{|1-x|}\leq \frac{(1/2)^{N+1}}{1/2}=\frac{1}{2^N}.$$

since $\lim_{N\to\infty}\frac{1}{2^N}=0$, $\lim_{N\to\infty}\max_E|f(x)-S_N(x)|=0$

& the partial sums $S_N(x)$ converge uniformly to $\frac{1}{1-x}$.

<u>Thm: Weierstrass M-Test</u> Let $\sum_{n=0}^{\infty} u_n(x)$ be a series of functions on a Domain E.
If there exist constants M_n, $n\geq 0$ s.t.

1) $\sum_{n=0}^{\infty} M_n$ converges

& 2) for $n\geq 0$, $|u_n(x)|\leq M_n$ for all x in E,

then $\sum_{n=0}^{\infty} u_n(x)$ converges absolutely for each x in E & is uniformly convergent on E.

eg $\sum_{n=1}^{\infty}\frac{x^n}{n^2}$: By Ratio Test, R.C. = 1:
$$\lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)^2}{x^n/n^2}\right| = |x|\lim_{n\to\infty}\left|\frac{n^2}{n^2+2n+1}\right| = |x| < 1$$

It also converges at the end points $x = \pm 1$:

$x = +1 \Rightarrow \sum_{n=\phi}^{\infty} \frac{1}{n^2}$ converges by p-test.

$x = -1 \Rightarrow \sum_{n=\phi}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely by p-test & thus converges.

Show that $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly on $[-1,1]$:

By M-test, $\left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2}$ for $-1 \le x \le 1$. $(M_n = 1/n^2)$

& $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test.

So $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly on $[-1,1]$.

eg $\sum_{n=1}^{\infty} \frac{\cos(nx)}{2^n}$ converges uniformly for all x:

$\left|\frac{\cos(nx)}{2^n}\right| \le \frac{1}{2^n}$ $(M_n = 1/2^n)$ for all x.

& $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges b/c it's geometric w/ $|r| = 1/2 < 1$.

So, by M-Test, $\sum_{n=1}^{\infty} \frac{\cos(nx)}{2^n}$ converges uniformly for all x.

<u>Properties of Uniformly convergent SERIES & SEQUENCES</u>.

<u>Thm</u> If $u_n(x)$ is continuous on E ~~and~~ for all $n \ge 0$, and
(Thm 31) $u(x) = \sum_{n=0}^{\infty} u_n(x)$ uniformly on E, then $u(x)$ is continuous on E.

<u>Thm</u> A uniformly convergent series of continuous functions on $[a,b]$
(Thm 32) can be integrated term by term

$$\int_a^b u(x)\,dx = \sum_{n=0}^{\infty}\left(\int_a^b u_n(x)\,dx\right)$$

<u>Thm</u> If $\sum_{n=0}^{\infty} u_n(x)$ converges to $f(x)$, for $a \le x \le b$, and if
(Thm 33) $\sum_{n=0}^{\infty} u_n'(x)$ converges uniformly for $a \le x \le b$, then
$\sum_{n=0}^{\infty} u_n'(x) = f'(x)$ for $a \le x \le b$.