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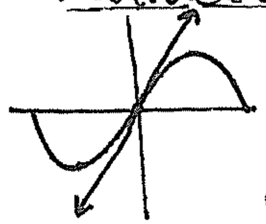
Sequences and Series

Motivation: $y=\sin x$ has tangent line $y=x$ at $(0,0)$



$y=x$ is a linear polynomial approximating $y=\sin x$ when x is close to 0.

Qu Can we get better approximations to $y=\sin x$ by taking higher degree polynomials?

yes. $y = x - \frac{x^3}{6}$ (degree 3), $y = x - \frac{x^3}{6} + \frac{x^5}{120}$ (degree 5)

And we can get arbitrarily good approximations by taking higher degree polynomials.
If we write out all terms that occur we get an expression for $\sin x$ as an "infinite polynomial"

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

This can also be done for any other function $y=f(x)$ that has infinitely many derivatives at $x=0$.
Goal: Make sense of these "infinite polynomials" and use them to study more complicated functions.

Sequences

Def A sequence of numbers is a list (often infinite) of real numbers $a_0, a_1, a_2, a_3, \ldots = \{a_n\}_{n\geq 0}$

eg • $1, 2, 3, 4, \ldots$

$a_n = n$ for $n \geq 1$. or $a_n = n+1$ for $n \geq 0$.

• $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$a_n = \frac{1}{n}$ for $n \geq 1$

• $1, -1, 1, -1, \ldots$

$$a_n = \begin{cases} 1, & \text{if } n \text{ even} \\ -1, & \text{if } n \text{ odd} \end{cases} \quad \text{for } n \geq 0$$

$$= (-1)^n = \cos(\pi n)$$

• $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$a_n = \frac{1}{2^n}$ $n \geq 0$

Geometric sequence

Def A <u>geometric sequence</u> has the form $a_n = a r^n$ for $n \geq 0$
where a = 1st term (a_0) and r = <u>constant ratio</u>:

<u>key property</u> for all $n \geq 0$

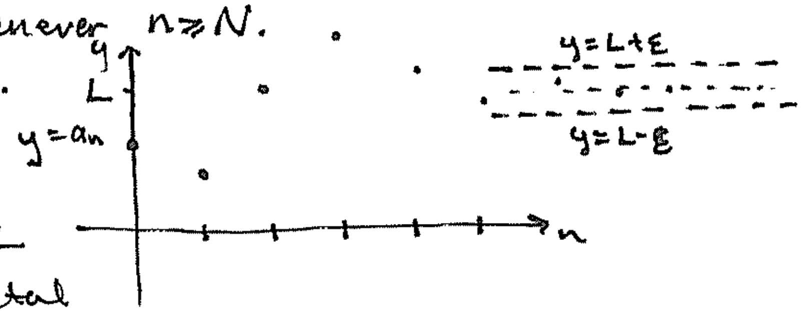
$$\frac{a_1}{a_0} = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{a_{n+1}}{a_n} = \cdots = r \quad \text{constant ratios}$$

<u>eg</u> • $3, 6, 12, 24, 48, 96, \ldots$ is geometric since all ratios are equal $\frac{96}{48} = 2 = \frac{48}{24} = \frac{24}{12} =$ etc.
$a_n = 3 \cdot 2^n$, $n \geq 0$

• $1, 3, 5, 7, 9, \ldots$ is not geometric since ratios are not equal $\frac{3}{1} = 3 \neq \frac{5}{3} \neq \frac{7}{5}$ etc.

<u>Note</u> The first term of a sequence may be a_0 or a_1 (or even a_{10}), so pay attention!

<u>Def</u> The sequence $\{a_n\}_{n \geq 0}$ <u>converges to a limit L</u> if for every $\varepsilon > 0$, there is an N such that $|a_n - L| < \varepsilon$ whenever $n \geq N$.
write $\lim_{n \to \infty} a_n = L$.



Pictorially, $\lim_{n \to \infty} a_n = L$ if for any horizontal strip between $y = L - \varepsilon$ and $y = L + \varepsilon$, the points of the sequence eventually lie inside it.

<u>eg</u> • $a_n = \frac{1}{n}$ $n \geq 1$: $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$.

• $a_n = n^2$ $n \geq 0$: $\lim_{n \to \infty} a_n = \lim_{n \to \infty} n^2 = \infty$

• $a_n = (-1)^n$ $n \geq 0$: $\lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^n =$ D.N.E.
values are ± 1 & never converge to anything.

<u>Thm</u> If $a_n = f(n)$ where $f(x)$ is a continuous function then $\lim_{n \to \infty} a_n = \lim_{n \to \infty} f(n) = \lim_{x \to \infty} f(x)$ if this limit exists.

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eg $a_n = \frac{2n^2-n}{3n^2+1}$: $\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{2x^2-x}{3x^2+1} = \frac{2}{3}$.

notice $\lim_{n\to\infty} a_n$ may exist, even if $\lim_{x\to\infty} f(x)$ doesn't.

eg $a_n = \sin(2\pi n)$, $n \geq 0$ but $\lim_{x\to\infty} \sin(2\pi x) =$ D.N.E.
$\Rightarrow a_n = 0$ for all n. b/c $\sin(2\pi x)$ oscillates.
$\Rightarrow \lim_{n\to\infty} a_n = 0$.

eg $a_n = \frac{n}{2^n}$: $\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{x}{2^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{2^x \cdot \ln 2} = \frac{1}{\infty} = 0$.

eg $a_n = (1+\frac{1}{n})^n$: $\lim_{n\to\infty} a_n = \lim_{x\to\infty} (1+\frac{1}{x})^x = y$.

$\ln y = \lim_{x\to\infty} \ln(1+\frac{1}{x})^x = \lim_{x\to\infty} x \cdot \ln(1+\frac{1}{x}) = \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}$

$\underset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+1/x}(-\frac{1}{x^2})}{(-1/x^2)} = \lim_{x\to\infty} \frac{1}{1+1/x} = 1$

$\Rightarrow y = e^1 = \boxed{e}$.

Infinite Series

Def An infinite series is an infinite sum $a_0 + a_1 + a_2 + a_3 + \cdots$
$= \sum_{n=0}^{\infty} a_n$ of an infinite sequence $\{a_n\}_{n\geq 0}$.

We can't actually add infinitely many things together, but notice: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ As we add more terms the sum gets closer & closer to 2.

$1, 3/2, 7/4, 15/8, 31/16, \ldots$

So it is reasonable to say that the infinite sum equals 2.

Complete Def Let $\{a_n\}_{n\geq 0}$ be a sequence. For $N \geq 0$, the N^{th} partial sum of $\{a_n\}_{n\geq 0}$ is the finite sum $S_N = a_0 + a_1 + \cdots + a_N = \sum_{j=0}^{N} a_j$. These partial sums form a sequence $\{S_N\}_{N\geq 0}$.

The Infinite Sum $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \sum_{j=0}^{N} a_j = \lim_{N\to\infty} S_N$.

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If the sequence $\{S_n\}_{n\geq 0}$ of partial sums has a limit, the series $\sum_{n=0}^{\infty} a_n$ <u>converges</u> to $\lim_{N\to\infty} S_N$.

If $\lim_{N\to\infty} S_N$ Does not exist (or is $\pm\infty$), then the series $\sum_{n=0}^{\infty} a_n$ <u>Diverges</u>.

eg. • $1+1+1+1+\cdots = \sum_{n=0}^{\infty} 1$ Diverges

since $S_N = \sum_{n=0}^{N} 1 = \underbrace{1+1+\cdots+1}_{N+1 \text{ times}} = N+1$

and $\lim_{N\to\infty} S_N = \lim_{N\to\infty} N+1 = \infty$.

• $1-1+1-1+1-1+\cdots = \sum_{n=0}^{\infty} (-1)^n$ Diverges

since $S_N = \sum_{n=0}^{N} (-1)^n = \underbrace{1-1+1-1+\cdots \pm 1}_{N+1} = \begin{cases} 1, & \text{if } N \text{ even} \\ 0, & \text{if } N \text{ odd} \end{cases}$

$\lim_{N\to\infty} S_N = \text{D.N.E.}$

• $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$ converges.

partial sums: $S_0 = 1$, $S_1 = 3/2$, $S_2 = 7/4$, $S_3 = 15/8$, ...

pattern: $S_n = \frac{2^{n+1}-1}{2^n} = 2 - \frac{1}{2^n}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(2 - \frac{1}{2^n}\right) = 2 - 0 = 2.$

<u>Geometric Series</u> If $\{a_n\}_{n\geq 0}$ is a geometric sequence, the series $\sum_{n=0}^{\infty} a_n$ is a <u>geometric series</u>.

In order to compute the sum, we need a formula for the sequence $\{S_N\}_{N\geq 0}$ of partial sums.

$a_n = a\cdot r^n \quad n \geq 0.$

$$
\begin{aligned}
S_N &= a_0 + a_1 + \cdots + a_N = a + ar + ar^2 + \cdots + ar^N \\
&= a(1 + r + r^2 + \cdots + r^N) \\
r S_N &= a(r + r^2 + r^3 + \cdots + r^N + r^{N+1}) \\
\hline
(1-r) S_N &= a(1 - r^{N+1})
\end{aligned}
$$

$$\Rightarrow \boxed{S_N = \frac{a(1-r^{N+1})}{1-r} = a + ar + ar^2 + \cdots + ar^N}$$