Partial Differential Equations

A **P.D.E.** is an equation relating a function of more than one variable to its derivatives. A **solution** is a function satisfying this equation (and any given **initial data**)

eg Given $\vec{V} = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$, is $\vec{V} = \text{grad}(F)$ for some function F?

$\vec{V} = \text{grad}(\vec{F}) \iff \frac{\partial F}{\partial x} = V_x,\ \frac{\partial F}{\partial y} = V_y,\ \frac{\partial F}{\partial z} = V_z$

$\iff$ F is a solution of the 3 P.D.E.'s

$$\frac{\partial u}{\partial x} = V_x,\quad \frac{\partial u}{\partial y} = V_y,\quad \frac{\partial u}{\partial z} = V_z.$$

eg Is $f(x,y,z)$ a Harmonic function?

$f(x,y,z)$ = Harmonic $\iff \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

$\iff$ f is a solution of the P.D.E.

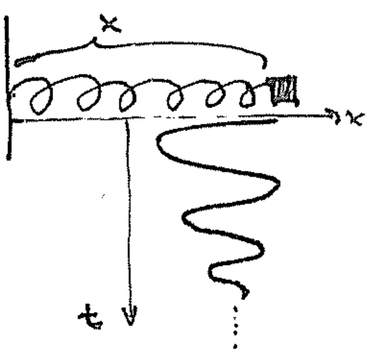
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

eg $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$. $\quad u = e^{-t}\sin x$ is a solution since

$\frac{\partial u}{\partial t} = -e^{-t}\sin x$

& $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^{-t}\cos x) = -e^{-t}\sin x. = \frac{\partial u}{\partial t}.$

eg Motion of a Mass on a Spring



x = position of mass = function of time t

$$m\frac{d^2x}{dt^2} + h\frac{dx}{dt} + k^2x = 0. \qquad (10.4)$$

m = mass, h = damping constant

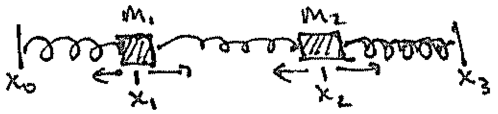
k^2 = spring constant.

If $h = 0$, the mass moves in <u>Simple Harmonic Motion</u>. The Solution

(10.7) is $x(t) = A\sin(\lambda t + \varepsilon)$ $\quad \lambda = k/\sqrt{m}$. = <u>frequency</u>

A = <u>amplitude</u> } constants determined
ε = <u>phase shift</u> } by initial position & velocity.

We can also consider 2 or more Masses connected by springs:



<u>2 Particles (Masses)</u>:

x_1 = position of mass m_1

x_2 = position of mass m_2

Let u_i = distance of x_i from its equilibrium position

$\Rightarrow$ u_1 & u_2 satisfy the O.D.E's

$$m_1 \frac{d^2 u_1}{dt^2} + h_1 \frac{du_1}{dt} - k^2(u_2 - 2u_1 + u_0) = F_1(t)$$

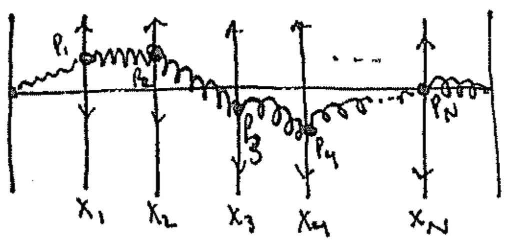
(10.25) outside Forces.

$$m_2 \frac{d^2 u_2}{dt^2} + h_2 \frac{du_2}{dt} - k^2(u_3 - 2u_2 + u_1) = F_2(t).$$

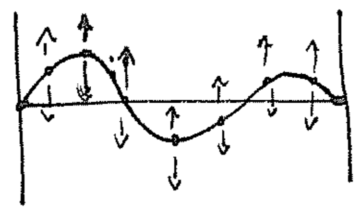
Different solutions depend on relations between the various constants. (Ch. 10.3)

<u>N Particles</u> A similar set up yields O.D.E's describing the horizontal motion of N masses connected by springs. (Ch. 10.4)

We may instead assume that each particle has a fixed x-value and moves vertically in the u-direction:

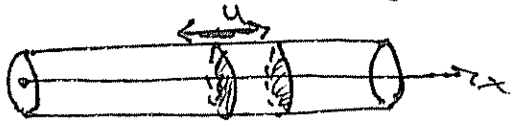


$N \to \infty$



as $N \to \infty$, the motion of the Particles represents the motion of a vibrating string (in xu-plane)

<u>or</u>, in terms of horizontal motion, we get a model for longitudinal vibrations of a Rod

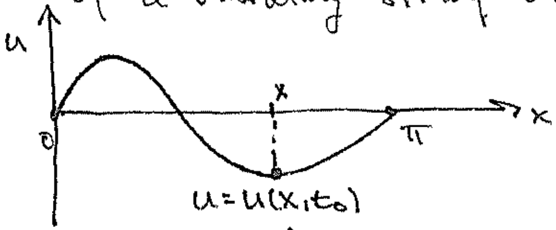


The WAVE EQUATION In One-Dimension

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 < x < \pi,\ t > 0$$

$$u(0,t) = u(\pi,t) = 0. \qquad a = \text{constant}$$

A solution $u(x,t)$ describes the position of a vibrating string at each time t.



$u = u(x,t_0)$

position of string at $t = t_0$

To find linearly independent solutions, called NORMAL MODES, let $u(x,t) = A(x)\sin(\lambda t + \epsilon)$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\Big(\lambda A(x)\cos(\lambda t+\epsilon)\Big) = -\lambda^2 A(x)\sin(\lambda t+\epsilon).$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\Big(A'(x)\sin(\lambda t+\epsilon)\Big) = A''(x)\sin(\lambda t+\epsilon).$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = \sin(\lambda t+\epsilon)\Big(-\lambda^2 A(x) - a^2 A''(x)\Big) = 0$$

$$\Rightarrow A''(x) = -\left(\frac{\lambda}{a}\right)^2 A(x)$$

This is an O.D.E w/ general solution

$$A(x) = c_1 \sin\left(\frac{\lambda}{a}x\right) + c_2\cos\left(\frac{\lambda}{a}x\right)$$

$u(0,t) = u(\pi,t) = 0$ for all $t \Rightarrow A(0) = A(\pi) = 0$.

$A(0) = c_1\cdot 0 + c_2\cos 0 = c_2 = 0 \quad \Rightarrow c_2 = 0.$

$A(\pi) = c_1\sin\left(\frac{\lambda}{a}\pi\right) = 0 \quad \Rightarrow \lambda/a = n \quad (n = 1,2,3,\dots)$

$\lambda_n = an$ = characteristic values

Resonant frequencies

~~⇒ Normal Modes~~ $\Rightarrow A_n(x) = \sin(nx) \qquad n = 1,2,3,\dots$

$\therefore \boxed{\text{Normal Modes} = \sin(nx)\sin(ant+\epsilon_n)} \qquad n = 1,2,3,\dots$

$\therefore$ <u>General Solution of Wave Equation</u> is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)\cdot\sin(ant+\varepsilon_n)}$$

provided we know this series and its 2nd derivatives converge, eg. $\boxed{\text{if } |c_n| < M/n^4 \text{ for all } n \geq 1}$

<u>other form</u>: $\sin(ant+\varepsilon_n) = \sin(ant)\cos(\varepsilon_n) + \cos(ant)\sin(\varepsilon_n)$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)\left[\cos(\varepsilon_n)\sin(ant) + \sin(\varepsilon_n)\cos(ant)\right]$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left[\alpha_n \sin(ant) + \beta_n \cos(ant)\right] \quad \alpha_n = c_n\cos(\varepsilon_n) \quad \beta_n = c_n \sin\varepsilon_n}$$

$u(x,0) = \sum_{n=1}^{\infty} \beta_n \sin(nx)$ = Initial Displacement of string.

$\Rightarrow$ $\boxed{\beta_n = \text{Fourier sine coefficients of initial displacement}}$ *

See p. 67 or §7.5

$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} na\alpha_n \sin(nx)$ = Initial velocity.

$\Rightarrow$ $\boxed{na\alpha_n = \text{Fourier sine coefficients of Initial Velocity.}}$ *

<u>Thm</u> Let $f(x)$, $g(x)$ be functions on $[0,\pi]$ s.t.

1) $f(x)$ has cont. derivatives thru order 4.
2) $g(x)$ has cont. derivatives thru order 3.
3) $f(0)=f(\pi)=f''(0)=f''(\pi)=0$
4) $g(0)=g(\pi)=g''(0)=g''(\pi)=0$

Then there exists a solution $u(x,t)$ of the wave equation s.t. $\boxed{u(x,0)=f(x)}$ (Initial displacement) & $\boxed{\frac{\partial u}{\partial t}(x,0)=g(x)}$ (Initial velocity)

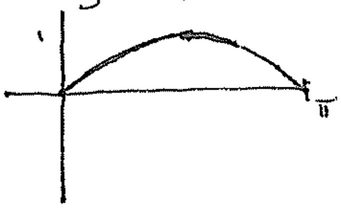
Namely, $\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left[\alpha_n \sin(ant) + \beta_n\cos(ant)\right]}$

w/ $\boxed{\alpha_n = \frac{2}{na\pi}\int_0^{\pi} g(x)\sin(nx)\,dx}$ * & $\boxed{\beta_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx}$ *

and this is the <u>unique</u> solution satisfying these initial conditions.

66

eg Suppose initial displacement is given by $f(x) = \sin x$



initial velocity $g(x) = 0$.

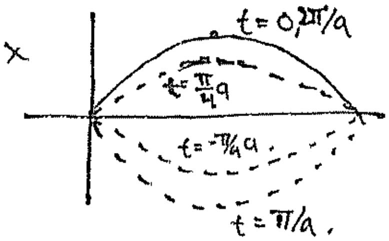
$\alpha_n = \frac{1}{na}$ (Fourier cosine coefficients of $g(x)$)
$= 0$.

β_n = Fourier sine coefficients of $f(x) = \sin x$
$= \begin{cases} 1, & n=1 \\ 0 & n \geq 2 \end{cases}$ since $\sin x = \sin x + 0 \ldots$ is a Fourier Series.

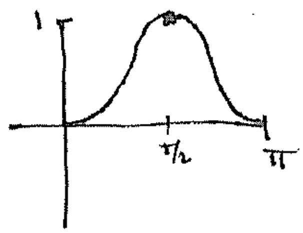
$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} [\sin(nx)\beta_n \cos(ant) + \sin(nx)\alpha_n \sin(ant)]$$

$$= \sin x \cdot \cos(at).$$

at $t = \pi/a$, $u(x, \pi/a) = \sin x \cos \pi = -\sin x$
at $t = 2\pi/a$, $u(x, 2\pi/a) = \sin x \cos 2\pi = \sin x$
at $t = \pi/4a$, $u(x, \pi/4a) = \sin x \cos \pi/4 = \frac{\sqrt{2}}{2} \sin x$



eg suppose $a=1$, $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$.
* Initial displacement is $f(x) = \sin^2 x$
* Initial velocity is $g(x) = 0$.



$g(x) = 0 \Rightarrow \alpha_n = 0$ (Fourier cosine coefficients of $\frac{1}{na} g(x) = 0$)

To find β_n we need the Fourier sine series of $f(x) = \sin^2 x$.

$$\beta_n = \frac{2}{\pi}\int_0^\pi \sin^2 x \sin(nx)\, dx$$

$$= \frac{2}{\pi}\left[\sin^2 x\left(-\frac{\cos(nx)}{n}\right)\Big]_0^\pi + \int_0^\pi 2\sin x \cos x \frac{\cos(nx)}{n}\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^\pi \frac{1}{n}\sin(2x)\cos(nx)\, dx\right]$$

$$= \frac{2}{n\pi}\int_0^\pi \frac{1}{2}\left(\sin((n+2)x) + \sin((2-n)x)\right) dx$$

$$= \frac{1}{n\pi}\left(-\frac{\cos(n+2)x}{n+2} - \frac{\cos(2-n)x}{2-n}\Big]_0^\pi\right) \quad (n \neq 2)$$

(If $n=2$, we still get 0)

$$= \frac{1}{n\pi}\left(\frac{-(-1)^{n+2}}{n+2} - \frac{(-1)^{2-n}}{2-n} + \frac{1}{n+2} + \frac{1}{2-n}\right) *$$

$$= \frac{1}{n\pi}\left((-1)^{n+1}\left(\frac{1}{n+2}+\frac{1}{2-n}\right)+\frac{1}{n+2}+\frac{1}{2-n}\right)$$

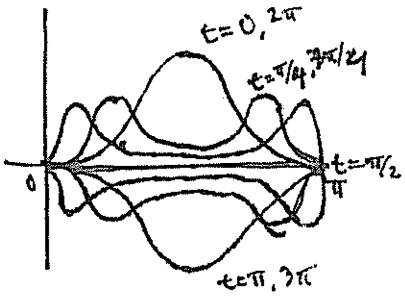
$$= \frac{1}{n\pi}\left(1+(-1)^{n+1}\right)\left(\frac{4}{4-n^2}\right) = -\frac{4(1-(-1)^n)}{n\pi(n^2-4)} = \begin{cases} 0, & n \text{ even} \\ \frac{-8}{n\pi(n^2-4)}, & n \text{ odd} \end{cases}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\cdot B_n \cos(nt)$$

$\underline{a=1.}$

$$= \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \sin(mx)\left(\frac{-8}{m(m^2-4)\pi}\right)\cos(mt)$$

$$\boxed{= -\sum_{k=1}^{\infty} \frac{8\sin((2k-1)x)\cos((2k-1)t)}{(2k-1)((2k-1)^2-4)\pi}}.$$



$u(x,0) = \sin^2 x = u(x,2\pi)$

$u(x,\pi) = -\sin^2 x$

since $\cos[(2k-1)\pi] = -1$.

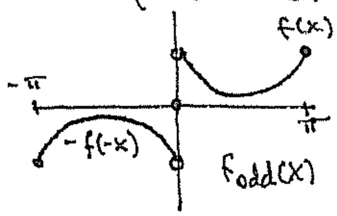
$u(x,\pi/2) = 0$

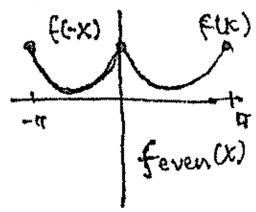
since $\cos((2k-1)\pi/2) = 0$.

NOTE: <u>Fourier Sine Series</u> & <u>Fourier Cosine Series</u> (Ch. 7.5)

If $f(x)$ is twice differentiable function on $[0,\pi]$ it can be extended to an odd function on $[-\pi,\pi]$

$$f_{odd}(x) = \begin{cases} f(x), & x>0 \\ 0, & x=0 \\ -f(-x), & x<0 \end{cases}$$





The Fourier series for $f_{odd}(x)$ has only sine terms, and it converges to $f(x)$ for $0<x<\pi$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{odd}(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -f(-x)\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx$$

$$\boxed{b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx} = \text{Fourier sine coefficients of } f(x)$$

$f(x)$ can also be extended to an even function on $[-\pi,\pi]$.

$f_{even}(x) = \begin{cases} f(x), & x\geq 0 \\ f(-x), & x<0 \end{cases}$. The Fourier series for $f_{even}(x)$ has only cosine terms & converges to $f(x)$ for $0<x<\pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{even}(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(-x)\cos(nx)\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$\Rightarrow \boxed{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx} \quad \text{Fourier Cosine coeff.'s of } f(x)$$