Solutions of the Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left[\alpha_n \sin(nat) + \beta_n \cos(nat)\right]$$

$$t=0 \Rightarrow f(x) = u(x,0) = \sum_{n=1}^{\infty} \beta_n \sin(nx).$$

$\Rightarrow \beta_n =$ Fourier sine coefficients for $f(x)$

$$\left(= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \right)$$

$$\Rightarrow \sum_{n=1}^{\infty} \beta_n \sin(nx)\cos(nat) = \sum_{n=1}^{\infty} \beta_n \left[\sin(n(x+at)) + \sin(n(x-at))\right]$$

$$= f(x+at) - f(x-at)$$

f was defined for $[0,\pi]$, If $x+at$ or $x-at$ is outside this interval, we have to use the odd-periodic extension of $f(x)$: $\tilde{f}(x) = \sum_{n=1}^{\infty} \beta_n \sin(nx)$.

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left[na\alpha_n \cos(nat) - \beta_n na \sin(nat)\right]$$

$$t=0 \Rightarrow g(x) = \sum_{n=1}^{\infty} \sin(nx)\, na\alpha_n$$

$\Rightarrow na\alpha_n =$ Fourier sine coefficients of $g(x)$.

$$\left(\alpha_n = \frac{2}{na\pi}\int_0^{\pi} g(x)\sin(nx)\,dx \right)$$

$$\Rightarrow \int g(x)\,dx = \sum_{n=1}^{\infty} -a\alpha_n \cos(nx) + C$$

$$\Rightarrow \sum_{n=1}^{\infty} \alpha_n \sin(nx)\sin(nat) = \frac{1}{2}\sum_{n=1}^{\infty} \alpha_n\left[\cos(n(x-at)) - \cos(n(x+at))\right]$$

$$= \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds.$$

where $g(x) = \sum_{n=1}^{\infty} \sin(nx)\, na\alpha_n$ is the odd periodic extension of $g(x)$ when x is outside $[0,\pi]$.

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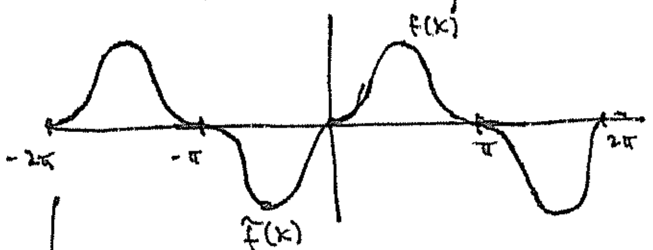
$\therefore\ u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left[\alpha_n \sin(nat) + \beta_n \cos(nat)\right]$

$= \frac{1}{2}\left[\tilde{f}(x+at) + \tilde{f}(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \tilde{g}(s)\, ds.$

where $\tilde{f}(x)$ = odd-periodic extension of Init. Disp. $f(x)$
$\tilde{g}(x)$ = odd-periodic extension of Init. vel. $g(x)$.

(If $f(x), g(x)$ are odd & periodic w/ period 2π, then
$\tilde{f}(x) = f(x)$ & $\tilde{g}(x) = g(x)$)

example from last time:
$f(x) = \sin^2 x$, $g(x) = 0$.



$\tilde{f}(x) = \begin{cases} \sin^2 x, & 2n\pi \le x \le (2n+1)\pi \\ -\sin^2 x, & (2n+1)\pi \le x \le (2n+2)\pi. \end{cases}$

$u(x,t) = \frac{1}{2}\left[\tilde{f}(x+at) + \tilde{f}(x-at)\right].$

$\underset{a=1}{=} \frac{1}{2}\left[\tilde{f}(x+t) + \tilde{f}(x-t)\right]$

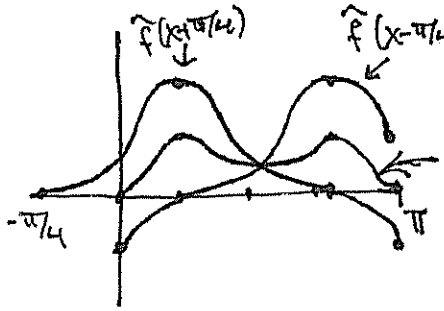
eg $f(x) = \sin^3 x$, $g(x) = -\sin 2x$. (both odd & periodic already)

$\Rightarrow u(x,t) = \frac{1}{2}\left[\sin^3(x+at) + \sin^3(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} -\sin 2s\, ds$

$= \boxed{\frac{1}{2}\left[\sin^3(x+at) + \sin^3(x-at)\right] + \frac{1}{4a}\left[\cos 2(x+at) - \cos 2(x-at)\right]}$

at $t = \pi/4$, $u(x, \pi/4) = \frac{1}{2}\left[\tilde{f}(x+\pi/4) + \tilde{f}(x-\pi/4)\right]$

shifted $\pi/4$ units to left. shifted $\pi/4$ units to right



$u(x, \pi/4) = \frac{\tilde{f}(x+\pi/4) + \tilde{f}(x-\pi/4)}{2}.$

Review:

Surface Integrals: S = Surface, oriented by normal $\vec{n}$.
parametrized by 3 functions
$x(u,v),\ y(u,v),\ z(u,v)$.

Different types of Integrals:

1) $$\iint_S f\, d\sigma = \iint_{R_{uv}} f(x(u,v), y(u,v), z(u,v)) \sqrt{EG-F^2}\, du\, dv.$$

2) $$\iint_S L\, dy\, dz + M\, dz\, dx + N\, dx\, dy = \iint_{R_{uv}} L\left|\frac{\partial(y,z)}{\partial(u,v)}\right| + M\left|\frac{\partial(z,x)}{\partial(u,v)}\right| + N\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

3) $$\iint_S \vec{F}\cdot d\vec{\sigma} = \iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint_S F_x\, dy\, dz + F_y\, dz\, dx + F_z\, dx\, dy$$

= Flux of $\vec{F}$ through S.

eg $S = \begin{cases} x = u+v & 0 \le u \le 1 \\ y = u\cos v & 0 \le v \le \pi \\ z = -u\sin v & \end{cases}$

$$\iint_S x\, dy\, dz + xy\, dz\, dx + z^2\, dx\, dy$$

$$= \int_0^\pi \int_0^1 (u+v)\begin{vmatrix} \cos v & -u\sin v \\ -\sin v & -u\cos v \end{vmatrix} + (u+v)u\cos v \begin{vmatrix} \sin v & -u\cos v \\ 1 & 1 \end{vmatrix}$$

$$+ u^2\sin^2 v \begin{vmatrix} 1 & 1 \\ \cos v & -u\sin v \end{vmatrix} \quad du\, dv \quad = \cdots$$

$$= \int_0^\pi \int_0^1 (u+v)(-u) + (u+v)u\cos v(u\cos v - \sin v) + u^2\sin^2 v(-u\sin v - \cos v)\, du\, dv.$$

Divergence Thm.

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_R \operatorname{div}(\vec{F})\, dx\, dy\, dz \qquad (\vec{n} = \text{outer normal})$$

only when S = entire Boundary of the 3-D region R.

eg S = unit sphere:

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_R \operatorname{div}(\vec{F})\, dx\, dy\, dz = \int_0^1 \int_0^\pi \int_0^{2\pi} \operatorname{div}(\vec{F})\, \rho^2 \sin\phi\, d\theta\, d\phi\, d\rho.$$

eg $$\iint_S 2z\, dy\, dz - xy\, dz\, dx + y^2\, dx\, dy = \iiint_R \operatorname{div}(2z\vec{i} - xy\vec{j} + y^2\vec{k})\, dx\, dy\, dz$$

$$= \iiint_R (-1)\, dx\, dy\, dz = \boxed{-1\cdot \mathrm{vol}(R)}.$$

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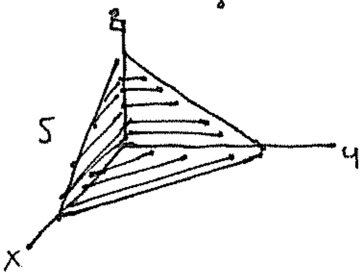
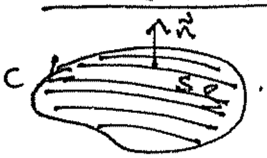
S = surface bounded by planes $x+y+z=1$, $z=0$, $y=0$, $x=0$.

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_R \operatorname{div}(\vec{F})\,dx\,dy\,dz$$



$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} \operatorname{div}(\vec{F})\,dz\,dy\,dx.$$

Stokes' Theorem C = simple closed curve
= boundary of a surface S.



$$\oint_C F_T\,ds = \oint_C F_x dx + F_y dy + F_z dz = \iint_S \operatorname{curl}(\vec{F})\cdot\vec{n}\,d\sigma$$

eg 
$C = \begin{cases} x = \cos(\pi t) \\ y = \sin(\pi t) \\ z = \sin(\pi t) + \cos^2(\pi t). \end{cases}$ = Boundary of $z = x^2+y$ over unit disk

$$\oint_C x^2\,dx - xy\,dy + yz\,dz = \iint_S \operatorname{curl}(x^2\vec{i} - xy\vec{j} + yz\vec{k})\cdot\vec{n}\,d\sigma$$

$$= \iint_S (z\vec{i} - y\vec{k})\cdot\vec{n}\,d\sigma$$

$$= \iint_{x^2+y^2\le 1} z\,dy\,dz - y\,dx\,dy$$

$$= \iint_{x^2+y^2\le 1} -2xz - y\;dx\,dy$$

$$= \iint_{x^2+y^2\le 1} -2x(x^2+y) - y\;dx\,dy\,. = \cdots$$

Series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}, \quad \sum_{n=1}^{\infty} 2^{1/n} \quad \sum_{n=1}^{\infty} n^{-n}$$

Taylor Series for $f(x) = \dfrac{1}{4+x} = \dfrac{1}{4(1+x/4)}$

$$= \frac{1}{5+(x-1)} = \frac{1}{5\left(1+\frac{(x-1)}{5}\right)}$$