(58) Integration + Differentiation of Fourier Series.

Thm Suppose $\sum_{n=0}^{\infty} u_n(x)$ converges uniformly to $u(x)$ on E, and each $u_n(x)$ is continuous on E. Then

1) $u(x)$ is continuous on E and
2) $\int u(x)\,dx = \sum_{n=0}^{\infty} \int u_n(x)\,dx$ (uniformly on E)

Cor If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$ on $[-\pi,\pi)$

then $\int f(x)\,dx = C + \frac{a_0}{2}x + \sum_{n=1}^{\infty}\left(-\frac{b_n}{n}\cos(nx) + \frac{a_n}{n}\sin(nx)\right)$

$$\therefore \boxed{\int f(x)\,dx = C + \sum_{n=1}^{\infty}\left(-\frac{b_n}{n}\cos(nx) + \left(\frac{a_n + a_0(-1)^{n+1}}{n}\right)\sin(nx)\right)}$$

since $x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$. This gives the Fourier series for $\int f(x)\,dx$.

and $C = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int f(x)\,dx\right)dx$

Recall: Fourier Series computed last time:

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx), \quad f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ -1, & -\pi \le x < 0 \end{cases} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)x)}{2n-1}$$

eg Find the Fourier Series for $f(x) = x^2$.

Integrate the Fourier Series for $2x = 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$

since $x^2 = \int 2x\,dx$.

$$= C + 4\sum_{n=1}^{\infty}\int\frac{(-1)^{n+1}}{n}\sin(nx)\,dx$$

$$= C + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

$$C = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\,dx = \frac{1}{2\pi}\left.\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right)$$

$$= \frac{1}{2\pi}\left(\frac{2\pi^3}{3}\right) = \frac{\pi^2}{3}.$$

eg Find the Fourier Series for $f(x) = x^3$.

Integrate the Fourier Series for $g(x) = 3x^2 = \pi^2 + 12\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$
since $x^3 = \int 3x^2\,dx$

$$= C + \pi^2 x + 12\sum_{n=1}^{\infty} \int \frac{(-1)^n}{n^2}\cos(nx)\,dx$$

$$= C + \pi^2 x + 12\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}\sin(nx)$$

$$= C + \pi^2 \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx) + 12\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}\sin(nx)$$

$$C = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3\,dx = \frac{1}{2\pi}\left(\frac{x^4}{4}\right)\Big]_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\frac{\pi^4}{4} - \frac{(-\pi)^4}{4}\right) = 0.$$

$$\therefore \boxed{x^3 = \sum_{n=1}^{\infty}\left(\frac{2\pi^2(-1)^{n+1}}{n} + \frac{12(-1)^n}{n^3}\right)\sin(nx)} \quad (-\pi < x < \pi)$$

eg Find the Fourier series for $|x|$:
Integrate the Fourier Series for $f(x) = \begin{cases} 1, & 0 < x \leq \pi \\ -1, & -\pi \leq x < 0 \end{cases}$
since $|x| = \int f(x)\,dx$

$$= C + \frac{4}{\pi}\sum_{n=1}^{\infty} \int \frac{\sin((2n-1)x)}{2n-1}\,dx$$

$$= C + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{-\cos((2n-1)x)}{(2n-1)^2}$$

$$C = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} x\,dx\right)$$

$$= \frac{1}{2\pi}\left(-\frac{x^2}{2}\Big]_{-\pi}^{0} + \frac{x^2}{2}\Big]_{0}^{\pi}\right) = \frac{1}{2\pi}(\pi^2) = \frac{\pi}{2}.$$

$$\Rightarrow \boxed{|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}, \quad -\pi < x < \pi}$$

Note: $y = |x|$, $y = x^2$ are even functions (y-axis symmetry) and their Fourier series have only cos terms.

~~Thm~~ Application: $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$

Let $x=0$: $0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$

$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}}$

Let $x=\pi$: $\cos(n\pi) = (-1)^n \Rightarrow (-1)^n\cos(\pi n) = (-1)^{2n} = 1.$

$\Rightarrow \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$

$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{3}/4 = \frac{\pi^2}{6}}$

Thm If $u(x) = \sum_{n=0}^{\infty} u_n(x)$ on E $(a \le x \le b)$ and the series of derivatives $\sum_{n=0}^{\infty} u_n'(x)$ converges uniformly on E, then it converges to $u'(x)$.

Cor If $f(x)$ has Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$ then $f'(x)$ has Fourier series

$$f'(x) = \sum_{n=1}^{\infty}(nb_n\cos(nx) - na_n\sin(nx))$$

Provided this series can be shown to converge uniformly on $[-\pi,\pi]$.

eg Suppose $f(x) = \sum_{n=1}^{\infty}\frac{\cos(nx)}{2^n}$. Then $f'(x) = \sum_{n=1}^{\infty}\frac{-n}{2^n}\sin(nx)$.

This series converges uniformly on $\mathbb{R}$ by M-test since

1) $\left|-\frac{n}{2^n}\sin(nx)\right| \le \frac{n}{2^n}$ and

2) $\sum_{n=1}^{\infty}\frac{n}{2^n}$ converges by Ratio test:

$$\lim_{n\to\infty}\left|\frac{(n+1)/2^{n+1}}{n/2^n}\right| = \lim_{n\to\infty}\left|\frac{2^n\cdot(n+1)}{2^{n+1}\cdot n}\right| = \lim\left|\frac{n+1}{2n}\right|$$

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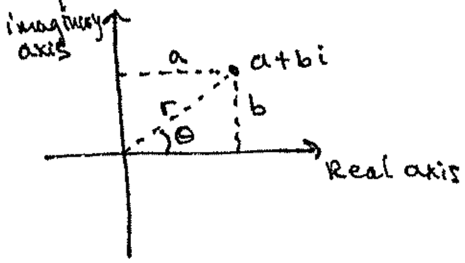
Complex Form of Fourier Series (7.17)

$e^{ix} = \cos x + i \sin x \quad (i^2 = -1,\ i = \sqrt{-1}) \quad$ for $x \in \mathbb{R}$.

(think of this as conversion between rectangular & polar coordinates in the complex plane:)



$$a+bi = r\cos\theta + i r \sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

$$\Rightarrow \cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

$$\Rightarrow \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n \sin(nx)\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{nix}+e^{-nix}\right) + \frac{b_n}{2i}\left(e^{nix}-e^{-nix}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - ib_n}{2}\right)e^{inx} + \left(\frac{a_n+ib_n}{2}\right)e^{-inx}\right]$$

$$= c_0 + \sum_{n=1}^{\infty}\left(c_n e^{inx} + d_n e^{-inx}\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where } c_0 = \frac{a_0}{2},\quad c_n = \begin{cases} \frac{a_n - ib_n}{2}, & n>0 \\ \frac{a_{-n}+ib_{-n}}{2}, & n<0. \end{cases}$$

are Complex numbers.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$

Pf
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos(nx) - i\sin(nx)\right)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx - \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \begin{cases} \frac{a_n - ib_n}{2}, & n>0 \\ a_0/2, & n=0 \\ \frac{a_n+ib_n}{2}, & n<0 \end{cases}$$

Complex Form of Fourier Series:

$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}}$ where $\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx}$