**Midterm 1 Solutions**

1. **[10 Points]** Evaluate $\int_C ydx + ydy + xdz$ where $C$ is the straight line path from $(3,-1,2)$ to $(2,1,-1)$. Does this integral depend on the path from $(3,-1,2)$ to $(2,1,-1)$? Explain.

   **Solution:** Parametrize the curve by $x = 3-t$, $y = -1+2t$, $z = 2-3t$ for $0 \leq t \leq 1$. So $dx = -dt$, $dy = 2dt$, and $dz = -3dt$. Therefore

   $$\int_C ydx + ydy + xdz = \int_0^1 (-1+2t)(-dt) + (-1+2t)(2dt) + (3-t)(-3dt)$$
   $$= \int_0^1 (5t-10)dt = \frac{-15}{2}$$

   The integral does depend on the path because $curl(y,y,x) = (0,-1,-1) \neq (0,0,0)$. Another way to argue this is to pick another path connecting the two points, evaluate the integral, and if you're lucky you'll get a different answer.

2. **[10 Points]**

   (a) Show that the vector field $v = (e^x siny,\ \ e^x cosy,\ \ 3)$ is irrotational.

   **Solution:** Calculate $\nabla \times v$.

   $$\nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x siny & e^x cosy & 3 \end{vmatrix} = 0i + 0j + 0k$$

   (b) Find a smooth function $f(x,y,z)$ so that $\nabla f = v$.

   **Solution:** $f(x,y,z) = e^x siny + 3z$

3. **[10 Points]** Verify Stokes' theorem for the vector field $v = (2y,\ \ 3x,\ \ -z^2)$, where $S$ is the upper half surface of the sphere $x^2+y^2+z^2 = 9$ and $C$ is its boundary.

   **Solution:** Parametrize $C$ by $x = 3cost$, $y = 3sint$, $z = 0$ for $0 \leq t \leq 2\pi$. Thus $dx = -3sintdt$, $dy = 3costdt$, and $dz = 0$. Thus

   $$\begin{aligned} \int_C 2ydx + 3xdy - z^2dz &= \int_0^{2\pi} \left(-18sin^2t + 27cos^2t\right) dt \\ &= \int_0^{2\pi} \left(-9 + 9cos(2t) + \tfrac{27}{2} + \tfrac{27}{2}cos(2t)\right) dt \\ &= 2\pi\left(-9 + \tfrac{27}{2}\right) \\ &= 9\pi \end{aligned}$$

Now use Stokes' Theorem: Parametrize $S$ by $\Phi(u,w) = (u, w, f(u,w))$ where $f(u,w) = \sqrt{1-u^2-w^2}$ and $(u,w)$ is in the disk $R_{uw}$ of radius 3 centered about $(0,0)$. Thus $P_1 = (1, 0, f_u)$ and $P_2 = (0, 1, f_w)$. So $P_1 \times P_2 = (-f_u, -f_w, 1)$. Also we have $curl(v) = (0,0,1)$. So $curl(v) \cdot (P_1 \times P_2) = 1$. By Stokes' Theorem, we have

$$\begin{aligned}
\iint_S curl(v) \cdot n d\sigma &= \iint_{R_{uw}} curl(v) \cdot (P_1 \times P_2)\, dudw \\
&= \iint_{R_{uw}} dudw \\
&= \texttt{area of disk of radius } 3 \\
&= 9\pi
\end{aligned}$$

4. **[10 Points]** Show that $\iint_S (\nabla \times v) \cdot n d\sigma = 0$ where $S$ is the boundary surface of a region $R$ in $\mathbb{R}^3$, $n$ is directed outward with respect to $R$, and $v$ is any smooth vector field defined on $\mathbb{R}^3$.

   **Solution:** By the divergence theorem,

$$\iint_S (\nabla \times v) \cdot n d\sigma = \iiint_R (\nabla \cdot (\nabla \times v))\, dxdydz$$

   For any smooth vector field $v$, the divergence of the curl of $v$ is zero. Therefore, the above integral is zero.

5. **[10 Points]** Let $v = (2xy + z,\ \ y^2,\ \ -(x+3y))$ be the velocity vector field, given in meters/second, of a fluid flowing in $\mathbb{R}^3$. What is the flow rate through the region bounded by the planes $x = 0$, $x = 3$, $y = 0$, $z = 0$, $y + z = 1$?

   **Solution:** The flow rate is the flux of $v$ through the boundary surface $S$ of the region $R$.

$$\texttt{flow rate} = \iint_S v \cdot n d\sigma$$

   By the divergence theorem, this is equal to $\iiint_R div(v) dxdydz$. The divergence of $v$ is $4y$. So the flow rate is

$$\begin{aligned}
\iiint_R 4y dxdydz &= \int_0^1 \int_0^{1-z} \int_0^3 4y dxdydz \\
&= \cdots \\
&= 2\ \texttt{meters}^3\texttt{/second}
\end{aligned}$$

6. **[10 Points]** Evaluate the integral

$$\oint_C \frac{y^3 dx - xy^2 dy}{\left(x^2+y^2\right)^2}$$

where $C$ is the square having vertices $(-1,-1)$, $(-1,1)$, $(1,1)$, $(1,-1)$ oriented counterclockwise. (Hint: $P_y = Q_x$.)

**Solutions:** Integrate instead around the circle of radius 1 centered about $(0,0)$. As a consequence of Green's Theorem, this is valid. Parametrize the circle by $x = \cos(t)$, $y = \sin(t)$ for $0 \leq t \leq 2\pi$. The integral becomes $\oint_{\texttt{circle}} \left(-\sin^4 t - \sin^2 t \cos^2 t\right) dt = \cdots = -\pi$