Math 5C Summer 2006 Midterm

Name: Answer key

Perm:

Your work must be neat and complete in order to receive full credit

When applying any theorems (like Green's or Stokes's) you must state that all the conditions for applying it are satisfied

All answers should be left in exact form; only exact decimals are fine. For instance, 1/10 and 0.1 are both acceptable, but 3.14 is not if the answer is π

If you need more space for work, use the back of the paper, but indicate on the front that there is work on the back.

You may find one of the following formulas helpful:

Polar: $\nabla^2 f(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$

Cylindrical: $\nabla^2 f(r,\theta,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$

Spherical: $\nabla^2 f(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}$

(1) (10 points) If S is the surface $z = 2, x^2 + y^2 \leq 1$, $\vec{n}$ is the unit upper normal, and $w = x^2y^2z$ which of the following are equivalent to $\iint_S \frac{\partial w}{\partial n} d\sigma$? (circle all that apply)

(a) $\iint_S (\nabla w) \quad \vec{n}\, d\sigma$

(b) $\iint_S |w\vec{n}|\, d\sigma$

(c) $\iint_S x^2y^2z\, d\sigma$

(d) $\iint_S 2x^2y^2\, d\sigma$

(e) $\iint_S \left(x^2y^2z\cos(\alpha) + x^2y^2z\cos(\beta) + x^2y^2z\cos(\gamma)\right) d\sigma$

(f) $\iint_S 2xy^2z\, dydz + 2x^2yz\, dzdx + x^2y^2\, dxdy$

(g) $\iint_{x^2+y^2\leq 1} x^2y^2\, dxdy$

(h) $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 2x^2y^2\, dydx$

(i) $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2y^2\, dxdy$

(j) $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 2x^2y^2\, dxdy$

(2) (10 points) Evaluate $\iint_S (\nabla f)\ \vec{n}\, d\sigma$ any way you choose where $f = 8xyz$, $\vec{n}$ is the unit <u>inner</u> normal, and S is the surface of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$

$\nabla f = (8yz, 8xz, 8xy)$

Since $\vec{n}$ is the inner normal, ~~and~~ S is the surface of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ and $\nabla \cdot (\nabla f) = 0 + 0 + 0 = 0$ is continuous we can apply the divergence theorem to get

$$\iint_S (\nabla f) \cdot \vec{n}\, d\sigma = -\iiint_R \nabla \cdot (\nabla f)\, dx\, dy\, dz$$ where

$$R = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$$

$$-\iiint_R 0\, dx\, dy\, dz = \boxed{0}$$

(3) (10 points) Consider $\int_C -3y\,dx + 3x\,dy + dz$ where C is the circle $x^2 + y^2 = 1$, $z = 2$ directed in the counterclockwise direction when looking from above.

(a) (5 points) Evaluate the integral by parameterizing the curve

$x = \cos(t) \quad y = \sin(t) \quad z = 2 \qquad 0 \le t \le 2\pi$

$dx = -\sin(t)\,dt \quad dy = \cos(t)\,dt \quad dz = 0\,dt$

$\int_0^{2\pi} \left(-3\sin(t)(-\sin(t)) + 3\cos(t)(\cos t) + 0\right) dt$

$= 3\int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt = 3\int_0^{2\pi} dt = \boxed{6\pi}$

(b) (5 points) Evaluate the integral by using Stokes's theorem

$\int_C -3y\,dx + 3x\,dy + dz = \int_C (-3y, 3x, 1) \cdot \vec{T}\,ds$

$\text{curl}(-3y, 3x, 1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3y & 3x & 1 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 6\vec{k}$

Since C is the boundary of the surface

$S = \{(x, y, z) ; x^2 + y^2 \le 1, z = 2\}$ and C is traced so that in order to apply Stokes's theorem $\vec{n}$ is the upper normal and $6\vec{k}$ is continuous we apply Stokes's thm to get

$\iint_S (0, 0, 6) \cdot \vec{n}\,d\sigma$ where $\vec{n} = \pm(0, 0, -1) = (0, 0, 1)$ ⇐ since upper normal.

$z = 2$, so $\iint_{x^2+y^2 \le 1} 6\sqrt{0^2 + 0^2 + 1}\,dx\,dy = 6 \cdot \text{area}(x^2 + y^2 \le 1) = \boxed{6\pi}$

(4) (10 points) The mantle of the Earth ranges from a radius of 3000 km to a radius of 6000 km It is known that the temperature at radius 3000 km is about 4000 C and at radius 6000 km the temperature is about 1000 C If the mantle is in temperature equilibrium find the temperature at any point in the mantle (Hint: There is some symmetry here)

The symmetry is spherical so we have $\nabla^2(T(R)) = 0$, where R is the radius from the center of the earth.

From the equation on the front if T depends only on R this reduces to:

$$\nabla^2(T(R)) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial T}{\partial R}\right) = 0 \Rightarrow \frac{1}{R^2} 2R \frac{\partial T}{\partial R} + \cancel{\frac{1}{R^2} R^2} \frac{\partial^2 T}{\partial R^2} = 0$$

$$\Rightarrow T'' + \frac{2T'}{R} = 0.$$ Letting $T' = y$ this gives:

$$y' + \frac{2y}{R} = 0 \quad \Rightarrow \int \frac{dy}{y} = \int -\frac{2}{R}\, dR \Rightarrow \ln|y| = -2\ln|R| + C_1$$

$$\Rightarrow |y| = e^{C_1} e^{-2\ln|R|} \Rightarrow |y| = \frac{e^{C_1}}{|R|^2} \Rightarrow y = \frac{\pm e^{C_1}}{R^2}$$

$$\Rightarrow y = \frac{C}{R^2} = CR^{-2} \Rightarrow T' = CR^{-2} \Rightarrow T = -CR^{-1} + D$$

$$T(3000) = \frac{-C}{3000} + D = 4000 \qquad \frac{18000000}{3000} + D = 4000 \Rightarrow D = -2000$$

$$- \quad T(6000) = \frac{-C}{6000} + D = 1000 \qquad \text{So } \boxed{T(R) = \frac{18{,}000{,}000}{R} - 2000}$$

$$\frac{2}{2}\ \frac{-C}{3000} - \frac{-C}{6000} = 3000 \qquad \frac{-C}{6000} = 3000 \Rightarrow -C = 18{,}000{,}000$$

(5) (10 points) For the three series, find if they converge or diverge and if it converges find the limit if possible

(a) $\sum_{n=4}^{\infty} \frac{n^2+1}{n^2-1}$

$$\lim_{n\to\infty} \frac{n^2+1}{n^2-1} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{1-\frac{1}{n^2}} = 1$$

$1 \neq 0$ so this $\boxed{\text{diverges}}$ by the n^{th} term test

(b) $\sum_{n=0}^{\infty} \frac{1}{n!}$

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0, \text{ so } \sum_{n=0}^{\infty} \frac{1}{n!} \ \boxed{\text{converges}}$$

by the ratio test. Although we do not yet know how to find this limit, it turns out to be $\boxed{e}$

(c) $\sum_{n=2}^{\infty} \left(\frac{n+1}{n+2} - \frac{n}{n+1} \right)$ This is almost identical to a homework question.

$$S_m = \sum_{n=2}^{m} \left(\frac{n+1}{n+2} - \frac{n}{n+1} \right) = \left(\frac{\cancel{3}}{\cancel{4}} - \frac{2}{3} \right) + \left(\frac{\cancel{4}}{\cancel{5}} - \frac{\cancel{3}}{\cancel{4}} \right) + \left(\frac{\cancel{5}}{\cancel{6}} - \frac{\cancel{4}}{\cancel{5}} \right) + \dots$$

$$+ \left(\frac{\cancel{m-2+1}}{\cancel{m-2+2}} - \frac{\cancel{m-2}}{\cancel{m-2+1}} \right) + \left(\frac{\cancel{m-1+1}}{\cancel{m-1+2}} - \frac{\cancel{m-1}}{\cancel{m-1+1}} \right) + \left(\frac{m+1}{m+2} - \frac{\cancel{m}}{\cancel{m+1}} \right)$$

$$= -\frac{2}{3} + \frac{m+1}{m+2} \quad \text{so} \quad \sum_{n=2}^{\infty} \left(\frac{n+1}{n+2} - \frac{n}{n+1} \right) = \lim_{m\to\infty} -\frac{2}{3} + \frac{m+1}{m+2}$$

$$= \lim_{m\to\infty} -\frac{2}{3} + \frac{1+\frac{1}{m}}{1+\frac{2}{m}} = -\frac{2}{3} + 1 = \boxed{\frac{1}{3}}$$

Extra Credit: Compute $\iint_S (\nabla \times \vec{v}) \cdot \vec{n}\, d\sigma$ where $\vec{v} = (y - z, yz, -xz)$, $\vec{n}$ is the outer normal and S consists of the 5 faces of the cube $0 \le x \le 2$, $0 \le y \le 2$, $0 \le z \le 2$ not in the xy-plane

Let S_1 be the surface which is all the faces of the cube. Since $\vec{V}$ is a polynomial, $\nabla \times \vec{V}$ and $\nabla \cdot (\nabla \times \vec{V})$ are continuous. Furthermore $\nabla \cdot (\nabla \times \vec{V}) = \vec{0}$ for any vector field. Since S_1 is the surface of the cube, and $\vec{n}$ is the outer normal and $\nabla \cdot (\nabla \times \vec{V})$ is continuous we can apply the divergence theorem to get

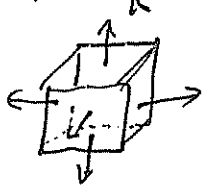
$$\iint_{S_1} (\nabla \times \vec{V}) \cdot \vec{n}\, d\sigma = \iiint_R 0 \, dx\, dy\, dz = 0$$

Let $S_2 = \{(x,y,z) : 0 \le x \le 2,\ 0 \le y \le 2,\ z = 0\}$

Then $0 = \iint_{S_1} (\nabla \times \vec{V}) \cdot \vec{n}\, d\sigma = \iint_S (\nabla \times \vec{V}) \cdot \vec{n}\, d\sigma + \iint_{S_2} (\nabla \times \vec{V}) \cdot \vec{n}\, d\sigma$

$$\Rightarrow \iint_S (\nabla \times \vec{V}) \cdot \vec{n}\, d\sigma = -\iint_{S_2} (\nabla \times \vec{V}) \cdot \vec{n}\, d\sigma$$ (where $\vec{n}$ is the outer normal)

$$\nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & yz & -xz \end{vmatrix} = -y\vec{i} + (z-1)\vec{j} - \vec{k}$$



~~Here the outer normal is~~ $\cancel{-\vec{k}}$

The outer normal for S_2 is $-\vec{k}$

$$-\iint_{S_2} (-y, z-1, -1) \cdot (0,0,-1)\, d\sigma = -\iint_{S_2} d\sigma = -(\text{Area of } S_2) = \boxed{-4}$$