Name: Solution.

Perm No.: ____________

Section Time :

Math 5C - Final Exam

March 21, 2007

Instructions:

- This exam consists of 9 problems, totaling 110 points, although it is out of 100 points.
- You must show all your work and fully justify your answers in order to recieve full credit. Partial credit will be given for work that is relevant and correct. You may leave your answers in unsimplified form, unless the problem asks you to simplify.
- No books or calculators are allowed. You may use one two-sided page of notes.
- Write your answers on the test itself, in the space alotted. You may attach additional pages if necessary.

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Total	

$$\sin A \cos B = \tfrac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\sin A \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \tfrac{1}{2}[\cos(A+B) + \cos(A-B)]$$

1. (10 pts) Let S be the surface consisting of the top half of the sphere $x^2 + y^2 + z^2 = 4$ of radius 2 ($z \geq 0$), and the disk $x^2 + y^2 = 4$ in the xy-plane, oriented with the outer normal $\mathbf{n}$. If $\mathbf{F}(x, y, z) = x^2\mathbf{i} - 2yz\mathbf{j} + z^2\mathbf{k}$, use the Divergence Theorem to express

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

as a triple integral in spherical coordinates. (DO NOT evaluate this integral!)

$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma = \iiint_R \operatorname{div}(x^2\vec{i} - 2yz\vec{j} + z^2\vec{k}) \, dx\,dy\,dz$$

$$= \iiint_R (2x - 2z + 2z) \, dx\,dy\,dz$$

$$= \iiint_R 2x \, dx\,dy\,dz$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \int_0^2 2\rho \cos\theta \sin\phi \, \rho^2 \sin\phi \, d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi} \int_0^2 2\rho^3 \cos\theta \sin^2\phi \, d\rho\,d\theta\,d\phi.$$

2. (15 pts) Use Stokes' Theorem to compute $\oint_C z^3 dy$, where C is the curve given by

$$\mathbf{r}(t) = \begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ z(t) = \cos t + \sin t \end{cases} \quad \text{for } 0 \le t \le 2\pi.$$

C is contained in the surface $z = x + y$. & the inside of C is above the interior of the unit circle $x = \cos t$, $y = \sin t$. Thus, C is the boundary of the surface S given by $z = x + y$. over the unit disk $x^2 + y^2 \le 1$ in the xy-plane.

By Stokes' Thm

$$\operatorname{curl}(z^3 \vec{j}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ D_x & D_y & D_z \\ 0 & z^3 & 0 \end{vmatrix} = -3z^2 \vec{i}$$

$$\oint_C z^3\,dy = \iint_S \operatorname{curl}(z^3 \vec{j}) \cdot \vec{n}\, d\sigma$$

$$= \iint_S -3z^2\, dy\, dz$$

$$= \iint_{x^2+y^2 \le 1} -3(x+y)^2 \left|\frac{\partial(y,z)}{\partial(x,y)}\right| dx\, dy \qquad \left|\frac{\partial(y,z)}{\partial(x,y)}\right| = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1.$$

$$= \iint_{x^2+y^2 \le 1} 3(x+y)^2\, dx\, dy$$

(S has upper normal since C is traversed counter clockwise)

$$= \int_0^{2\pi} \int_0^1 3(r^2 + 2r^2 \cos\theta \sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{3r^4}{4}\Big]_0^1 (1 + 2\cos\theta \sin\theta)\, d\theta$$

$$= \frac{3}{4}(\theta + \sin^2\theta)\Big]_0^{2\pi} = \frac{3(2\pi)}{4} = \boxed{\frac{3\pi}{2}}$$

3. (15 pts) Do the series below converge or diverge? Justify your answers.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n-1}$. converges by Alternating Series test

Since $b_n = \frac{1}{3n-1} > 0$ and

1) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{3n-1} = 0$

& 2) $b_n = \frac{1}{3n-1} > b_{n+1} = \frac{1}{3n+2}$ for all $n \geq 1$.

(b) $\sum_{n=1}^{\infty} 2^{1/n}$.

$\lim_{n\to\infty} 2^{1/n} = 2^0 = 1 \neq 0$ so it Diverges by the n^{th} Term Test.

(c) $\sum_{n=1}^{\infty} \frac{2n}{n^3+n-1}$.

For $n \geq 1$, $n^3+n-1 \geq n^3$.

Thus, $\frac{2n}{n^3+n-1} \leq \frac{2n}{n^3} = \frac{2}{n^2}$.

$\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges by p-series test.

Thus, by the comparison test, $\sum_{n=1}^{\infty} \frac{2n}{n^3+n-1}$ converges too.

4. (10 pts) Determine all values of x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^{nx}}$ converges.

Using the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n+1/2^{(n+1)x}}{n/2^{nx}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cdot 2^{nx}}{n\cdot 2^{nx}\cdot 2^{x}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n\cdot 2^{x}} \right| = \frac{1}{2^{x}} < 1.$$

$\Rightarrow 2^{x} > 1$ in order for the series to converge.

$\Rightarrow x\cdot \ln 2 > \ln 1 = 0$

$\Rightarrow x > 0$.

we check whether it converges at $x=0$:

$\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^{n\cdot 0}} = \sum_{n=1}^{\infty} n$ diverges by the n^{th} Term test since

$\displaystyle\lim_{n\to\infty} n = \infty \neq 0$.

Thus, the series converges for all x in $\boxed{(0,\infty)}$.

5. (10 pts) Show that the series $\sum_{n=0}^{\infty} \frac{(\ln x)^n}{n+1}$ converges uniformly on the interval $[1, \sqrt{e}]$.

We use the M-test.

$|u_n(x)| = \left|\frac{(\ln x)^n}{n+1}\right| \leq \left|\frac{(\ln \sqrt{e})^n}{n+1}\right|$ for $1 \leq x \leq \sqrt{e}$

$= \frac{(1/2)^n}{n+1} = \frac{1}{2^n(n+1)}$.

So let $M_n = \frac{1}{2^n(n+1)}$.

$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \frac{1}{2^n(n+1)}$ converges ~~[illegible]~~

since $\frac{1}{2^n(n+1)} \leq \frac{1}{2^n}$

and $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges b/c it is geometric w/ $|r| = 1/2 < 1$.

Thus $\sum_{n=0}^{\infty} M_n$ converges by the comparison test

& $\therefore \sum_{n=0}^{\infty} \frac{(\ln x)^n}{n+1}$ converges uniformly by the M-test.

6. (15 pts) Find the Maclaurin series for $f(x) = \sin^2 x$. (Hint: Use the identity $2 \sin x \cos x = \sin(2x)$.)

$$f'(x) = 2 \sin x \cos x = \sin(2x).$$

Since $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1}$$

$$\Rightarrow f(x) = \int \sin(2x)\,dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} \left(\frac{x^{2n+2}}{2n+2}\right) + C$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+2)!} x^{2n+2}$$

plug in $x = 0$: $\sin^2 0 = C = 0$.

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+2)!} x^{2n+2}$$ or, letting $k = n+1$,

$$= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k-1} 2^{2k-1}}{(2k)!} x^{2k}}$$

7. (10 pts) Find the Fourier series for the function $f(x) = \begin{cases} 1, & \text{if } -\pi \le x \le 0 \\ 0, & \text{if } \quad 0 < x \le \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} dx = \frac{1}{\pi}(0 - -\pi) = 1.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\cos(nx)\,dx \qquad (n>0)$$

$$= \frac{1}{\pi}\left.\frac{\sin(nx)}{n}\right]_{-\pi}^{0} = \frac{1}{n\pi}(\sin 0 - \sin(-n\pi))$$

$$= 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\sin(nx)\,dx$$

$$= \frac{-1}{\pi}\left(\left.\frac{\cos(nx)}{n}\right)\right]_{-\pi}^{0} = -\frac{1}{\pi}\left(\frac{\cos 0}{n} - \frac{\cos(-n\pi)}{n}\right)$$

$$= -\frac{1}{\pi}\left(\frac{1}{n} - \frac{(-1)^n}{n}\right)$$

$$= \frac{1}{n\pi}((-1)^n - 1)$$

$$\Rightarrow f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n\pi}\sin(nx)$$

$$= \boxed{\frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin((2k-1)x)}{2k-1}}$$

8. (10 pts) Use the previous question (or other methods) to find the Fourier series for

$$g(x) = \begin{cases} \pi + x, & \text{if } -\pi \le x \le 0 \\ \pi, & \text{if } \quad 0 < x \le \pi \end{cases}.$$

$$g'(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases} = f(x) \text{ from previous problem.}$$

$$\Rightarrow g'(x) \sim \frac{1}{2} + \frac{-2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

$$\Rightarrow g(x) \sim \int \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}\,dx$$

$$= C + \frac{1}{2}x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

$$C = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} \pi + x\,dx + \int_0^{\pi} \pi\,dx\right)$$

$$= \frac{1}{2\pi}\left(\left(\pi x + \frac{x^2}{2}\right)\Big]_{-\pi}^{0} + \pi x\Big]_0^{\pi}\right)$$

$$= \frac{1}{2\pi}\left(-\pi(-\pi) - \frac{(-\pi)^2}{2} + \pi^2\right)$$

$$= \frac{1}{2\pi}\left(\frac{3\pi^2}{2}\right) = \frac{3\pi}{4}.$$

From class $x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx)$

$$\Rightarrow g(x) \sim \frac{3\pi}{4} + \frac{1}{2}\left(2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx)\right) + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

$$= \boxed{\frac{3\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{n}\sin(nx) + \frac{2}{\pi(2n-1)^2}\cos(2n-1)x\right]}$$

9. (15 pts) Find the solution $u(x,t)$ of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - 9\frac{\partial^2 u}{\partial x^2} = 0,$$

subject to the boundary conditions $u(0,t) = u(\pi,t) = 0$, and with initial displacement and velocity given by

$$u(x,0) = \sin x - 2\sin(3x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 3\sin(2x).$$

The general solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\left[\alpha_n \sin(nat) + \beta_n \cos(nat)\right]$$

where B_n = n^{th} Fourier Sine coefficient of Init. Disp.

= coefficient of $\sin(nx)$ in Fourier Sine series of $\sin x - 2\sin(3x)$

$\Rightarrow \beta_1 = 1,\ \beta_3 = -2,\ \beta_n = 0$ for all other n.

$\alpha_n = (\text{Coefficient of } \sin(nx) \text{ in Init. Velocity})/na$.

I.V. $= 3\sin(2x)$

$\Rightarrow \alpha_2 = \frac{3}{2a} = \frac{3}{2\cdot 3} = \frac{1}{2}$; $\alpha_n = 0$ for all other n.

Since $a = \sqrt{9} = 3$, the series for $u(x,t)$

becomes:

$$u(x,t) = \sin x \cdot \beta_1 \cos(3t) + \sin(2x)\cdot \alpha_2 \sin(6t) + \sin(3x)\cdot \beta_3 \cos(9t)$$

$$= \boxed{\sin x \cdot \cos(3t) + \frac{1}{2}\sin(2x)\sin(6t) - 2\sin(3x)\cos(9t)}$$