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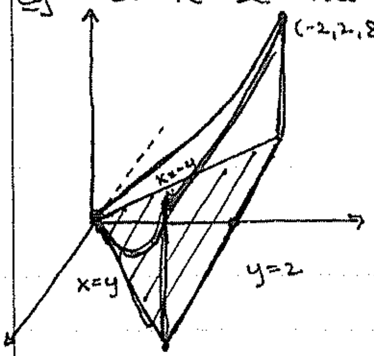
HW p.121 6a

Triple Integrals

eg Let R be the region $-y \le x \le y,\ 0 \le y \le 2,\ 0 \le z \le x^2+y^2$



a) find Volume of R

b) Find Mass of R if the density at (x,y,z) is $\rho(x,y,z) = 1+y-x$

a) $$V = \iiint_R 1\,dx\,dy\,dz = \int_0^2 \int_{-y}^{y} \int_0^{x^2+y^2} dz\,dx\,dy$$

$$= \int_0^2 \int_{-y}^{y} (x^2+y^2)\,dx\,dy = \int_0^2 \left(\frac{x^3}{3} + y^2x\right)\Big]_{-y}^{y} dy$$

$$= \int_0^2 \frac{y^3}{3} + y^3 - \frac{-y^3}{3} - -y^3\,dy = \int_0^2 \frac{8}{3}y^3\,dy = \frac{2}{3}y^4\Big]_0^2 = \boxed{\frac{32}{3}}$$

b) $$M = \iiint_R \rho(x,y,z)\,dx\,dy\,dz = \int_0^2 \int_{-y}^{y} \int_0^{x^2+y^2} 1+y-x\,dz\,dx\,dy$$

$$= \int_0^2 \int_{-y}^{y} (1+y-x)(x^2+y^2)\,dx\,dy = \int_0^2 \int_{-y}^{y} (1+y)y^2 - y^2x + (1+y)x^2 - x^3\,dx\,dy$$

$$= \int_0^2 (1+y)y^2(y - -y) - y^2\left(\frac{y^2}{2} - \frac{y^2}{2}\right) + \frac{(1+y)}{3}(y^3 - -y^3) - \left(\frac{y^4}{4} - \frac{y^4}{4}\right) dy$$

$$= \int_0^2 2y^3(1+y) + \frac{2}{3}y^3(1+y)\,dy = \int_0^2 \frac{8}{3}(y^3+y^4)\,dy = \frac{8}{3}\left(\frac{y^4}{4} + \frac{y^5}{5}\right)\Big]_0^2$$

$$= \frac{8}{3}\left(\frac{52}{5}\right) = \boxed{\frac{416}{15}}$$

eg Calculate the Volume of a sphere of radius r.

$R = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le r^2\}$.

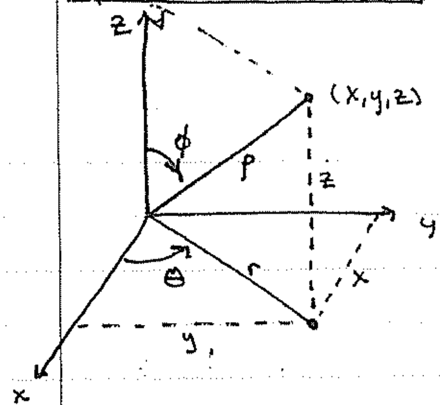
$-r \le x \le r,\ -\sqrt{r^2-x^2} \le y \le \sqrt{r^2-x^2},\ -\sqrt{r^2-x^2-y^2} \le z \le \sqrt{r^2-x^2-y^2}$.

$$V = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} dz\,dy\,dx$$

$$= \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 2\sqrt{r^2-x^2-y^2}\,dy\,dx$$

To avoid a complicated trig substitution, we should transform the original integral into Spherical coordinates.

⑥ Spherical Coordinates



$(x,y,z) \Rightarrow (\rho, \theta, \phi) \qquad 0 \le \rho < \infty,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$

$z = \rho\cos\phi \quad , \quad r = \rho\sin\phi$

$x = r\cos\theta = \rho\cos\theta\sin\phi$

$y = r\sin\theta = \rho\sin\theta\sin\phi$

$\rho = \sqrt{x^2+y^2+z^2}$

S = solid sphere of radius R, centered at Origin:

$0 \le \rho \le R,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$

To transform $\iiint_S dx\,dy\,dz$ into spherical coordinates, we must multiply by the Jacobian determinant.

$$dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| d\rho\, d\theta\, d\phi :$$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{vmatrix} = \rho^2\sin\phi$$

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$$\therefore V = \int_0^R\int_0^{2\pi}\int_0^\pi \rho^2\sin\phi\, d\phi\, d\theta\, d\rho = \int_0^R\int_0^{2\pi} -\rho^2\cos\phi\Big]_0^\pi d\theta\, d\rho$$

$$= \int_0^R\int_0^{2\pi} 2\rho^2\, d\theta\, d\rho = \int_0^R 4\pi\rho^2\, d\rho = \frac{4\pi}{3}\rho^3\Big]_0^R = \boxed{\frac{4}{3}\pi R^3}.$$

In general, to change an integral into spherical coordinates, express R in terms of inequalities of ρ, θ, ϕ ∋ use the formula:

$$\boxed{\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R f(\rho\cos\theta\sin\phi,\ \rho\sin\theta\sin\phi,\ \rho\cos\phi)\ \rho^2\sin\phi\, d\rho\, d\theta\, d\phi}$$

eg $\iiint_R e^{(x^2+y^2+z^2)^{3/2}}\,dx\,dy\,dz$ where R is the portion of the unit sphere w/ $x, y, z \ge 0$.

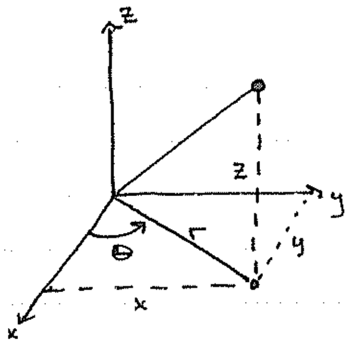
$$= \int_0^1\int_0^{\pi/2}\int_0^{\pi/2} e^{\rho^3}\rho^2\sin\phi\, d\phi\, d\theta\, d\rho = \int_0^1\int_0^{\pi/2} \rho^2 e^{\rho^3}(-\cos\phi)\Big]_0^{\pi/2} d\theta\, d\rho$$

$$= \int_0^1\int_0^{\pi/2} \rho^2 e^{\rho^3}\, d\theta\, d\rho = \frac{\pi}{2}\int_0^1 \rho^2 e^{\rho^3}\, d\rho = \frac{\pi}{2}\cdot\frac{1}{3}e^{\rho^3}\Big]_0^1 = \boxed{\frac{\pi}{6}(e-1)}$$

⑦ <u>Cylindrical Coordinates</u>

polar coordinates in xy-plane + z-coordinate

$(x,y,z) \longmapsto (r,\theta,z)$



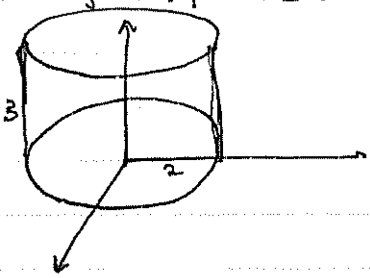
$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ r &= \sqrt{x^2+y^2} \end{aligned}}$$

$$dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dr\,d\theta\,dz = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} dr\,d\theta\,dz = r\,dr\,d\theta\,dz.$$

$\Rightarrow$ To transform an integral into cylindrical coordinates, express R in terms of inequalities of r, θ, z and use the formula:

$$\boxed{\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz}$$

eg $\displaystyle\iiint_R e^{x^2+y^2-z}\,dx\,dy\,dz \qquad R = \{(x,y,z) \mid x^2+y^2 \le 4,\ 0 \le z \le 3\}.$



$$= \int_0^3 \int_0^{2\pi} \int_0^2 e^{r^2} e^{-z}\, r\,dr\,d\theta\,dz$$

$$= \int_0^3 \int_0^{2\pi} \frac{1}{2} e^{r^2}\Big]_0^2 e^{-z}\,d\theta\,dz$$

$$= \int_0^3 \int_0^{2\pi} \frac{1}{2}(e^4-1)e^{-z}\,d\theta\,dz = \int_0^3 \pi(e^4-1)e^{-z}\,dz$$

$$= -\pi(e^4-1)e^{-z}\Big]_0^3 = \boxed{-\pi(e^4-1)\left(\frac{1}{e^3}-1\right)}.$$

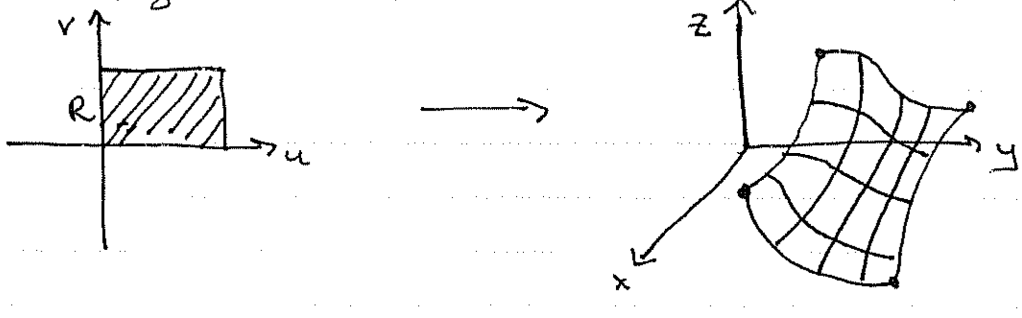
eg $\iiint_R z\,dx\,dy\,dz \qquad R$ = region bounded by cylinder $x^2+y^2=1$, xy-plane on bottom & cone $z=\sqrt{x^2+y^2}$ on top.

$$= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{x^2+y^2}} z\, r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}(x^2+y^2)r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{2} r^3\,dr\,d\theta = \int_0^{2\pi} \frac{1}{8}\,d\theta = \boxed{\frac{\pi}{4}}$$

Parametrized Surfaces:

A surface S in $\mathbb{R}^3$ can be thought of as a mapping of a region in the plane into space



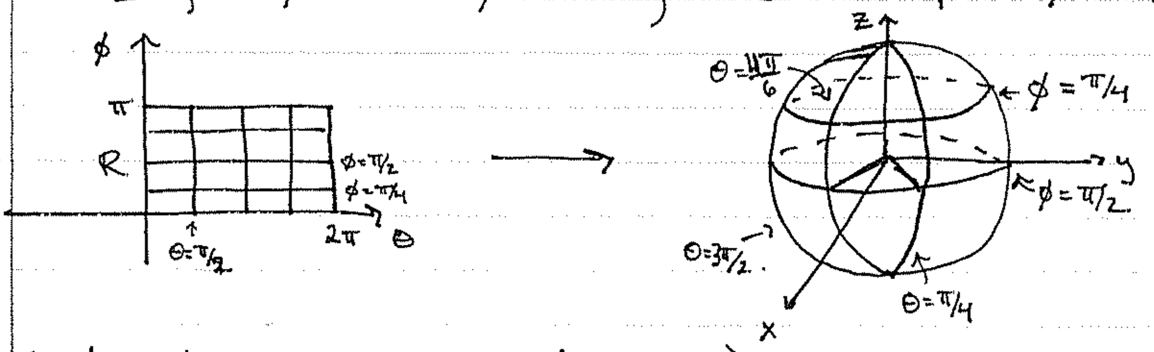
Such a mapping is specified by 3 equations:

$$x = x(u,v)$$
$$y = y(u,v) \qquad (u,v) \text{ is in } R \quad \left(\text{eg } \begin{matrix} a \le u \le b \\ f(u) \le v \le g(u) \end{matrix}\right)$$
$$z = z(u,v)$$

(We assume that the map $R \longrightarrow S$ is one-to-one)

eg Unit sphere: In spherical coordinates, the unit sphere is given by the simple equation $\rho = 1$.

So
$$\left.\begin{aligned} x &= \rho\sin\phi\cos\theta = \sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta = \sin\phi\sin\theta \\ z &= \rho\cos\phi = \cos\phi \end{aligned}\right\} \begin{matrix}\text{parametrization of unit sphere} \\ 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi. \end{matrix}$$



(not quite 1-to-1, but good enough)

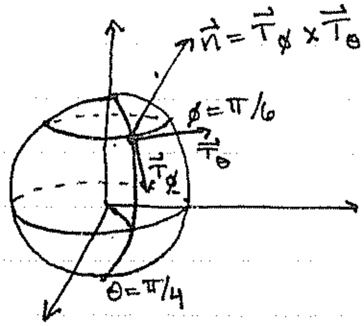
Let's use this parametrization to find the normal vector to S at $(\theta, \phi) = (\frac{\pi}{4}, \pi/6)$.

First find the tangent vectors in the θ & ϕ directions.

$$\vec{T}_\theta = x_\theta \vec{i} + y_\theta \vec{j} + z_\theta \vec{k} = -\sin\tfrac{\pi}{6}\sin\tfrac{\pi}{4}\,\vec{i} + \sin\tfrac{\pi}{6}\cos\tfrac{\pi}{4}\,\vec{j} + 0\vec{k}$$
$$= -\tfrac{\sqrt{2}}{4}\vec{i} + \tfrac{\sqrt{2}}{4}\vec{j}$$

$$\vec{T}_\phi = x_\phi \vec{i} + y_\phi \vec{j} + z_\phi \vec{k} = \cos(\pi/6)\cos\tfrac{\pi}{4}\,\vec{i} + \cos(\tfrac{\pi}{6})\sin(\tfrac{\pi}{4})\,\vec{j} + -\sin(\pi/6)\,\vec{k}$$

⑨



$$\vec{T}_\theta = -\frac{\sqrt{2}}{4}\vec{i} + \frac{\sqrt{2}}{4}\vec{j}$$

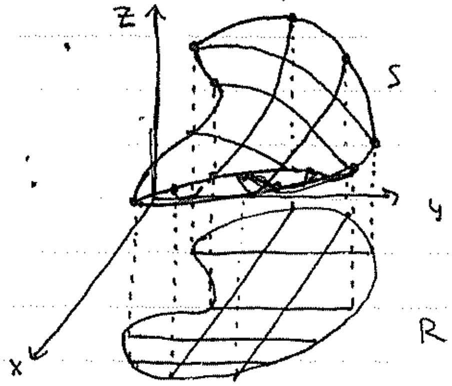
$$\vec{T}_\phi = \frac{\sqrt{6}}{4}\vec{i} + \frac{\sqrt{6}}{4}\vec{j} - \frac{1}{2}\vec{k}$$

$$\vec{n} = \vec{T}_\phi \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & -\frac{1}{2} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0 \end{vmatrix}$$

$$= \frac{\sqrt{2}}{8}\vec{i} + \frac{\sqrt{2}}{8}\vec{j} + \frac{\sqrt{12}}{8}\vec{k}$$ (or any scalar multiple)

eg If S has equation $z = f(x,y)$, it can be parameterized using x & y as parameters.

$$S = \begin{cases} x = x \\ y = y \\ z = f(x,y) \end{cases} \quad (x,y) \text{ is in } R$$



$z = f(x,y)$ over a region R in xy-plane.

Recall: the normal vector to S at a point P is obtained using the gradient of $F(x,y,z) = z - f(x,y) = 0$.

$\nabla F = (-f_x, -f_y, 1)$

$\vec{n}_p = \nabla F(p) = (-f_x(p), -f_y(p), 1)$

HW Exercises 1) Find the Mass of a ball of radius 1 (centered at the origin) with density function

$$\delta(x,y,z) = \frac{1}{1+x^2+y^2+z^2}$$

2) Use Cylindrical Coordinates to find the Volume of the region bounded by the xy-plane & the surface $z = 4 - x^2 - y^2$.