(53) <u>Fourier Series</u>

We have seen that functions $f(x)$ with continuous derivatives of all orders near $x=a$, have a Taylor Series expansion at $x=a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The partial sums of this series are (finite) <u>polynomials</u> that give better & better approximations of $f(x)$.

We now try to find approximations to $f(x)$ using trigonometric functions: $\sin(nx)$ & $\cos(nx)$ $n \geq 0$.

<u>Def</u> A <u>trigonometric series</u> is a series of the form

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right) \quad \text{for } a_n, b_n \in \mathbb{R}.$$

$$= \tfrac{1}{2}a_0 + a_1\cos x + b_1\sin x + a_2\cos(2x) + b_2\sin(2x) + \cdots$$

<u>eg</u> $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}$, $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^2}$, $\displaystyle 1+\sum_{n=1}^{\infty}\frac{2^n\sin(nx)+n^2\cos(nx)}{n!}$

are examples of trigonometric series.

<u>OBSERVATIONS</u> Assume that $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$ (that is, the trig. series converges to $f(x)$). Then

- $f(x)$ is <u>periodic</u> w/ period $= 2\pi$.
  $\therefore f(x+2\pi) = f(x)$ for all $x$.
  - this is because the functions $\cos(nx)$ & $\sin(nx)$ all have a period of $2\pi$:
    $\cos(n(x+2\pi)) = \cos(nx)$, $\sin(n(x+2\pi)) = \sin(nx)$

  (In fact $\cos(nx), \sin(nx)$ have period $2\pi/n$)

- $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n \sin(nx+\alpha)$ = <u>Amplitude-Phase form</u>

where $\boxed{A_n = \sqrt{a_n^2 + b_n^2}, \quad a_n = A_n\sin\alpha, \quad b_n = A_n\cos\alpha}$

$A_n$ = <u>Amplitude</u>. (vertical height of sine wave)

$\alpha$ = <u>phase shift</u> (Amount sine wave is shifted to left)

Pf: Let $A_n = \sqrt{a_n^2 + b_n^2}$ : $A_n \sin(nx + \alpha) = A_n \sin(nx)\cos\alpha + A_n \cos(nx)\sin\alpha$

$$= \underline{A_n \cos\alpha} \sin(nx) + \underline{A_n \sin\alpha} \cos(nx)$$
$$= b_n \sin(nx) + a_n \cos(nx)$$
$$\Rightarrow a_n = A_n \sin\alpha \ \ \& \ \ b_n = A_n \cos\alpha.$$

So $\alpha = \sin^{-1}(a_n/A_n) = \cos^{-1}(b_n/A_n)$

which is well-defined since $\left| a_n/\sqrt{a_n^2+b_n^2} \right| \le 1$.

__Problem__ Given $f(x)$ how can we find $a_n$ & $b_n$ s.t.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(nx) + b_n \sin(nx) \right] \ ?$$

__Thm__ If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(nx) + b_n \sin(nx) \right]$, then

a) $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$, for $n \ge 0$

b) $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$, for $n \ge 1$.

__proof__ we verify a):

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\, dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos(mx) + \sum_{n=1}^{\infty} \left[ a_n \cos(nx)\cos(mx) + b_n \sin(nx)\cos(mx) \right] dx$$

$$= \int_{-\pi}^{\pi} \frac{a_0}{2} \cos(mx)\, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos(nx)\cos(mx)\, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin(nx)\cos(mx)\, dx.$$

$$\cos A \cos B = \tfrac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$
$$\sin A \cos B = \tfrac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$
$$\sin A \sin B = \tfrac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$$

$$\Rightarrow \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)x)\, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)x)\, dx$$

$$= \frac{1}{2} \frac{\sin((n+m)x)}{n+m}\Big]_{-\pi}^{\pi} + \frac{1}{2} \frac{\sin((n-m)x)}{n-m}\Big]_{-\pi}^{\pi}$$

$= 0$ if $n - m \ne 0$. ie. if $n \ne m$

or $= \frac{1}{2}\int_{-\pi}^{\pi} \cos 0\, dx = \frac{1}{2}\int_{-\pi}^{\pi} dx = \pi$ if $n - m = 0$, ie if $n = m$.

and $\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\sin((n+m)x)\,dx + \frac{1}{2}\int_{-\pi}^{\pi}\sin((n-m)x)\,dx$

$$= \frac{1}{2}\frac{(-\cos((n+m)x))}{n+m}\Big]_{-\pi}^{\pi} + \frac{1}{2}\,\frac{\sin((n-m)x)}{n-m}\Big]_{-\pi}^{\pi}$$

$= 0$ if $n \neq m$.

or $= 0 + \frac{1}{2}\int_{-\pi}^{\pi}\sin(0)\,dx = 0 + \frac{1}{2}\int_{-\pi}^{\pi} 0\,dx = 0$ if $n = m$.

$$\therefore \int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mx)\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin(nx)\cos(mx)\,dx$$

$$= \begin{cases} \pi a_0, & \text{if } m = 0. \\ \pi a_n, & \text{if } m = n. \end{cases} = \pi a_m.$$

b) is proved similarly.

<u>Def</u> For $f(x)$ continuous on $[-\pi, \pi]$ (except possibly at a finite # of points), the <u>Fourier Series</u> for $f(x)$ is the trigonometric series

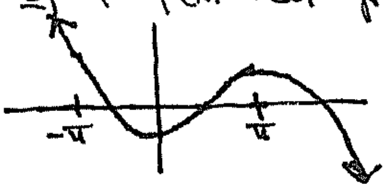
$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$ for $n \geq 0$

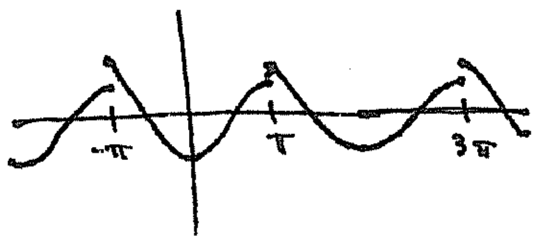
& $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$ for $n \geq 1$.

~~Thm~~ <u>Rmk</u> If $f(x)$ is not periodic of period $2\pi$, the function its Fourier series converges to <u>will</u> be.

<u>eg</u> if $f(x)$ has graph




F.S. of $f(x)$ =



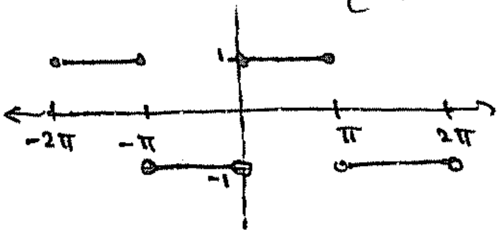
Theorem I If $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))$ converges uniformly to $f(x)$, then $f(x)$ is continuous for all $x$, has period $2\pi$, and the series is the Fourier series of $f(x)$.

Cor If 2 trigonometric series converge to the same function $f(x)$, then ~~all~~ they have the same coefficients:

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(nx)+b_n\sin(nx) = \frac{a_0'}{2}+\sum_{n=1}^{\infty} a_n'\cos(nx)+b_n'\sin(nx)$$

for all $x \implies a_0=a_0'$, $a_n=a_n'$ & $b_n=b_n'$ for all $n\geq 1$.

eg Let $f(x)=\begin{cases}1, & 0\leq x\leq \pi\\ -1, & -\pi\leq x<0\end{cases}$ = "square wave" when extended to a periodic function.




$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}-\cos(nx)\,dx+\frac{1}{\pi}\int_{0}^{\pi}\cos(nx)\,dx$$

If $n=0$: $=\frac{1}{\pi}\int_{-\pi}^{0}-1\,dx+\frac{1}{\pi}\int_{0}^{\pi}1\,dx=\frac{1}{\pi}(-\pi)+\frac{1}{\pi}(\pi)=0$

If $n\neq 0$: $=\frac{1}{\pi}\left.\frac{-\sin(nx)}{n}\right]_{-\pi}^{0}+\frac{1}{\pi}\left.\frac{\sin(nx)}{n}\right]_{0}^{\pi}=0.$

$a_n=0$ for $n\geq 0$.

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)\,dx=\frac{1}{\pi}\int_{-\pi}^{0}-\sin(nx)\,dx+\frac{1}{\pi}\int_{0}^{\pi}\sin(nx)\,dx$$

$$=\frac{1}{\pi}\left.\frac{\cos(nx)}{n}\right]_{-\pi}^{0}+\frac{1}{\pi}\left.\frac{(-\cos(nx))}{n}\right]_{0}^{\pi}$$

$$=\frac{1}{\pi}\left(\frac{1}{n}-\frac{(-1)^n}{n}\right)-\frac{1}{\pi}\left(\frac{(-1)^n}{n}-\frac{1}{n}\right)=\frac{2}{n\pi}-\frac{(-1)^n 2}{n\pi}$$

$$=\begin{cases}0, & n=2,4,6,\ldots\\ \frac{4}{n\pi}, & n=1,3,5,\ldots\end{cases}$$

$$\Rightarrow f(x)=\frac{4}{\pi}\sin x+\frac{4}{3\pi}\sin(3x)+\frac{4}{5\pi}\sin(5x)+\cdots$$

$$=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)x)}{2n-1}$$

(see fig. 7.5 in text)
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eg $f(x) = x$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = \frac{1}{\pi}\left(\frac{x}{n}\sin(nx)\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin(nx)}{n}\,dx\right) \quad (n \neq 0)$$

$u = x \quad v = \frac{1}{n}\sin(nx)$

$du = dx \quad dv = \cos(nx)dx$

$$= \frac{1}{\pi}\left(\frac{\pi}{n}\cdot 0 - \frac{(-\pi)}{n}\cdot 0 + \frac{\cos(nx)}{n^2}\Big]_{-\pi}^{\pi}\right) = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\frac{x^2}{2}\Big]_{-\pi}^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{1}{\pi}\left(-\frac{x}{n}\cos(nx)\Big]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nx)}{n}\,dx\right)$$

$u = x \quad v = -\frac{1}{n}\cos(nx)$

$du = dx \quad dv = \sin(nx)dx$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi}{n}(-1)^n - \frac{+\pi}{n}(-1)^n\right) + \frac{\sin(nx)}{n^2}\Big]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left(-\frac{2\pi}{n}(-1)^n + 0 - 0\right)$$

$$= -\frac{2}{n}(-1)^n.$$

$$\Rightarrow \boxed{x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)}$$

<u>Convergence of Fourier Series</u> – <u>Fundamental Theorem</u>

Let $f(x)$ be <u>piecewise very smooth</u> on $[-\pi, \pi]$ (this means that $f(x)$ is made up of pieces on subintervals of $[-\pi, \pi]$ that are continuous w/ continuous 1st & 2nd derivatives). Then the Fourier Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right) \quad \text{converges to } f(x)$$

wherever $f(x)$ is continuous inside $[-\pi, \pi]$.

at discontinuities: $x = x_1$

$$\text{F.S.} \longrightarrow \frac{1}{2}\left[\lim_{x\to x_1^-} f(x) + \lim_{x\to x_1^+} f(x)\right]$$

at $x = \pm\pi$, $\text{F.S.} \longrightarrow \frac{1}{2}\left[\lim_{x\to -\pi^+} f(x) + \lim_{x\to \pi^-} f(x)\right]$

The convergence is uniform in any closed interval where $f(x)$ is continuous.