Name: Solution

Perm No.: ____________

Section Time :

Math 5C - Midterm 1

February 6, 2007

Instructions:

- This exam consists of 5 problems worth 10 points each, for a total of 50 possible points.
- You must show all your work and fully justify your answers in order to recieve full credit. You may leave your answers in unsimplified form, unless the problem asks you to simplify.
- No books or calculators are allowed. You may use a one-sided page of notes.
- Write your answers on the test itself, in the space alotted. You may attach additional pages if necessary.

1	
2	
3	
4	
5	
Total	

1. (a) Compute $\int_C 3x^2\,dx - 2xyz\,dy + xe^{yz}\,dz$ where C is the straight line path from $(0,0,0)$ to $(2,1,-1)$.

C has Parametric Equations:

$$\begin{cases} x = 2t \\ y = t \\ z = -t \end{cases} \quad 0 \le t \le 1 \quad \Rightarrow \int_C 3x^2\,dx - 2xyz\,dy + xe^{yz}\,dz$$

$$= \int_0^1 3(2t)^2(2) - 2(2t)(t)(-t)\cdot 1 + 2t\,e^{-t^2}(-1)\,dt$$

$$= \int_0^1 24t^2 + 4t^3 - 2te^{-t^2}\,dt = 8t^3 + t^4 + e^{-t^2}\Big]_0^1$$

$$= 8 + 1 + e^{-1} - 0 - 0 - 1$$

$$= \boxed{8 + e^{-1}}$$

(b) Is the integral in (a) independent of path? Justify your answer.

The integral can be written as $\int_C u_T\,ds$

where $\vec{u} = 3x^2\vec{i} - 2xyz\vec{j} + xe^{yz}\vec{k}$.

Since all of these functions are differentiable on $\mathbb{R}^3$ the integral is path-independent $\Longleftrightarrow$ $\text{curl}(\vec{u}) = \vec{0}$.

But $\text{curl}(\vec{u}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & -2xyz & xe^{yz} \end{vmatrix} = (xze^{yz} + 2xy)\vec{i} + \cdots$

$\neq \vec{0}$

Thus the integral is $\boxed{\text{not path-independent}}$

2. Find the surface area of the surface S, which is parametrized by

$$\phi(u,v) = \begin{cases} x(u,v) = u\cos v \\ y(u,v) = u\sin v \\ z(u,v) = 1-u^2 \end{cases}$$

for all (u,v) with $0 \le u \le 1$ and $0 \le v \le 2\pi$.

$$S.A. = \iint_{R_{uv}} \sqrt{EG-F^2}\, du\, dv$$

where $E = x_u^2 + y_u^2 + z_u^2 = \cos^2 v + \sin^2 v + (-2u)^2 = 1 + 4u^2$

$G = x_v^2 + y_v^2 + z_v^2 = u^2\sin^2 v + u^2\cos^2 v + 0 = u^2$

$F = x_u x_v + y_u y_v + z_u z_v = (\cos v)u(-\sin v) + (\sin v)u\cos v + (-2u)\cdot 0$

$= -u\cos v\sin v + u\cos v \sin v = 0.$

$$\Rightarrow S.A. = \int_0^{2\pi}\int_0^1 \sqrt{(1+4u^2)u^2}\, du\, dv$$

$$= \int_0^{2\pi}\int_0^1 u\sqrt{1+4u^2}\, du\, dv$$

let $w = 1+4u^2$

$dw = 8u\, du$

$\frac{1}{8}dw = u\, du$

$$= \int_0^{2\pi}\int_{w=1}^{5} \frac{1}{8}\sqrt{w}\, dw\, dv$$

$$= \int_0^{2\pi} \frac{1}{8}\cdot\frac{2}{3} w^{3/2}\Big]_1^5\, dv$$

$$= \int_0^{2\pi} \frac{1}{12}(5\sqrt{5}-1)\, dv = \boxed{\frac{\pi}{6}(5\sqrt{5}-1)}$$

3. Let S be the sphere $x^2 + y^2 + z^2 = 9$ of radius 3, oriented with the inner normal $\mathbf{n}$. Compute

$$\int\int_S x^2\, dy\, dz + y^2\, dz\, dx + z^2\, dx\, dy.$$

If $\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$,

$$\iint_S x^2\, dy\, dx + y^2\, dz\, dx + z^2\, dx\, dy$$

$$= \iint_S \vec{F} \cdot \vec{n}\, d\sigma = - \iiint_R \operatorname{div}(\vec{F})\, dx\, dy\, dz$$

where R is the ~~unit~~ radius 3 - Ball $x^2 + y^2 + z^2 \le 9$ and we need the minus sign since S is oriented by the <u>inner</u> normal.

$$= \iiint_R (2x + 2y + 2z)\, dx\, dy\, dz$$

$$= 2\int_0^{\pi} \int_0^{2\pi} \int_0^3 (\rho \cos\theta \sin\phi + \rho \sin\theta \sin\phi + \rho \cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= 2 \int_0^{\pi} \int_0^{2\pi} \int_0^3 \rho^3 (\cos\theta \sin^2\phi + \sin\theta \sin^2\phi + \cos\phi \sin\phi)\, d\rho\, d\theta\, d\phi$$

$$= 2 \int_0^{\pi} \int_0^{2\pi} \frac{3^4}{4} (\cos\theta \sin^2\phi + \sin\theta \sin^2\phi + \cos\phi \sin\phi)\, d\theta\, d\phi$$

$$= \frac{81}{2} \int_0^{\pi} \sin^2\phi \underbrace{(\sin\theta)\Big]_0^{2\pi}}_{0} + \sin^2\phi \underbrace{(-\cos\theta)\Big]_0^{2\pi}}_{0} + 2\pi \cos\phi \sin\phi\, d\phi$$

$$= \frac{81}{2} \int_0^{\pi} 2\pi \cos\phi \sin\phi\, d\phi = 81\pi \left(\frac{\sin^2\phi}{2} \right) \Big]_0^{\pi} = 81\pi (0 - 0) = \boxed{0}$$

4. Compute $\oint_C 2xz\,dx - 2yz\,dy + x\,dz$, where C is the curve given by

$$\mathbf{r}(t) = \begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ z(t) = \sin(2t) + \sin t \end{cases}$$

for $0 \leq t \leq 2\pi$.

We use Stokes' Theorem.

The curve C lies on the surface $z = 2xy + y$

since $z(t) = \sin(2t) + \sin t = 2\cos t \sin t + \sin t$

$= 2\,x(t)\,y(t) + y(t)$.

Thus C is the boundary of the surface S given by the graph of $z = 2xy + y$ over the unit disk $x^2 + y^2 \leq 1$ (Since the unit disk is the interior of the curve $x = \cos t$, $y = \sin t$ that is directly under C on the xy-plane)

If $\vec{F} = 2xz\,\vec{i} - 2yz\,\vec{j} + x\,\vec{k}$, $\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ D_x & D_y & D_z \\ 2xz & -2yz & x \end{vmatrix}$

$= 2y\,\vec{i} + (2x-1)\,\vec{j} + 0\,\vec{k}$

$$\Rightarrow \oint_C F_T\,ds = \iint_S 2y\,dy\,dz + (2x-1)\,dz\,dx$$

$$= \iint_{x^2+y^2 \leq 1} \left(-2y f_x - (2x-1) f_y\right) dx\,dy \qquad \text{where } f(x,y) = z = 2xy + y$$

$$= \iint_{x^2+y^2\leq 1} -2y(2y) - (2x-1)(2x+1)\,dx\,dy = \iint_{x^2+y^2\leq 1} 1 - 4(x^2+y^2)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 (1-4r^2)\,r\,dr\,d\theta = 5\int_0^{2\pi} \left[\frac{r^2}{2} - r^4\right]_0^1 d\theta = \int_0^{2\pi} -\frac{1}{2}\,d\theta = \boxed{-\pi}$$

5. Let S be the top half of the unit sphere (i.e., S is given by $x^2+y^2+z^2=1$ and $z \geq 0$), oriented by the outer normal $\mathbf{n}$, and let $\mathbf{F}$ be the vector field

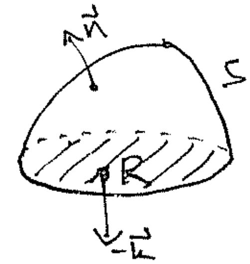
$$\mathbf{F}(x,y,z) = (x+ye^z)\mathbf{i} + e^{xz}\mathbf{j} + (x^2+y^2)\mathbf{k}.$$

Integrate $\int\int_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$.

If R is the unit disk $x^2+y^2 \leq 1$ in the xy-plane then R and S together form the entire boundary of the 3-dimensional region $H: x^2+y^2+z^2 \leq 1$ and $z \geq 0$.
ie, the top half of the unit ball.

Since S is oriented by the outer normal, R should be as well, and thus on R, we choose $\vec{n} = -\vec{k}$

By the Divergence Theorem:



$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma + \iint_R \vec{F}\cdot(-\vec{k})\,d\sigma = \iiint_H \operatorname{div}(\vec{F})\,dx\,dy\,dz$$

$$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_R \vec{F}\cdot\vec{k}\,d\sigma + \iiint_H \operatorname{div}(\vec{F})\,dx\,dy\,dz$$

$$= \iint_R x^2+y^2\,dx\,dy + \iiint_H 1\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^1 r^2\cdot r\,dr\,d\theta + \operatorname{Vol}(H)$$

$$= \int_0^{2\pi}\frac{1}{4}\,d\theta + \operatorname{Vol}(H) = \frac{\pi}{2} + \frac{1}{2}\left(\frac{4}{3}\pi(1)^3\right)$$

$$= \frac{\pi}{2} + \frac{2}{3}\pi = \boxed{\frac{7\pi}{6}}$$