Name: Solutions

Perm No.: ____________

Section Time :

# Math 5C - Midterm 2

March 1, 2007

## Instructions:

- This exam consists of 5 problems worth 10 points each, for a total of 50 possible points.
- You must show all your work and fully justify your answers in order to recieve full credit. You may leave your answers in unsimplified form, unless the problem asks you to simplify.
- No books or calculators are allowed. You may use a one-sided page of notes.
- Write your answers on the test itself, in the space alotted. You may attach additional pages if necessary.

| | |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| Total | |

1. Consider the sequence $\{a_n\}_{n\geq 0}$ given by

$$4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, \ldots$$

(a) Find a formula for the $n^{th}$ term $a_n$ of this sequence in terms of $n$, starting with $n = 0$.

It is a geometric sequence, since

$$\frac{8/3}{4} = \frac{2}{3} = \frac{16/9}{8/3} = \frac{32/27}{16/9} \quad \text{∋} \quad r = \frac{2}{3}.$$

$a = 1^{st}$ term $= a_0 = 4$.

$$\boxed{a_n = ar^n = 4(2/3)^n \quad \text{for } n \geq 0}$$

(b) Does the sequence $\{a_n\}_{n\geq 0}$ converge or diverge? If it converges, find its limit.

Since $|r| = 2/3 < 1$ it $\boxed{\text{converges to } 0}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 4(2/3)^n = \boxed{0}$$

(c) Find the sum of the series $\sum_{n=0}^{\infty} a_n$ or show that it diverges.

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} 4(2/3)^n = \frac{a}{1-r}$$

$$= \frac{4}{1-2/3}$$

$$= \frac{4}{1/3}$$

$$= \boxed{12}$$

2. Do the series below converge or diverge? Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

So the series $\boxed{\text{Diverges}}$ by the $n^{th}$ term test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

$$a_n = \frac{1}{\sqrt{n^3+1}} \leq \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a p-series with $p = 3/2 > 1$,

so it converges.

Therefore, by the comparison test

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} \ \boxed{\text{converges}}$$

3. Find the radius and interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{2^n(x-1)^n}{2n-1}$$

Ratio Test: to find R.C.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-1)^{n+1}/(2(n+1)-1)}{2^n(x-1)^n/(2n-1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2(x-1)\cdot(2n-1)}{2n+1}\right|$$

$$= |x-1| \lim_{n\to\infty}\left|\frac{4n-2}{2n+1}\right|$$

$$= 2|x-1| < 1$$

$\Rightarrow$ we need $|x-1| < 1/2$ for the series to converge. Therefore, $\boxed{R = 1/2}$

The endpoints of the interval of convergence are $a \pm R = 1 \pm 1/2 = 1/2,\ 3/2$.

If $x = 1/2$: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(-1/2)^n}{2n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n-1}$ is an Alternating Series (for $n \geq 1$)

For $n \geq 1$, let $b_n = \frac{1}{2n-1} > 0$. Since 1) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n-1} = 0$.

and 2) $b_n = \frac{1}{2n-1} > \frac{1}{2n+1} = b_{n+1}$ for all $n \geq 1$,

the series converges by the Alternating Series test.

If $x = 3/2$: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(1/2)^n}{2n-1} = \sum_{n=0}^{\infty} \frac{1}{2n-1}$ Diverges by comparison Test

Since $\frac{1}{2n-1} > \frac{1}{2n}$ for $n \geq 1$ & $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

$\therefore \boxed{\text{I.C.} = [1/2, 3/2)}$

4. Find the Taylor series for $f(x) = \dfrac{1}{\sqrt{x}}$, centered at $x = 1$, and determine its radius of convergence.

(For full credit, your answer should be written in $\Sigma$-notation. To receive partial credit, you may write out the first five terms of the Taylor series.)

$$f(x) = \frac{1}{\sqrt{x}} = x^{-1/2} \quad \Rightarrow \quad f(1) = 1$$

$$f'(x) = -\frac{1}{2}x^{-3/2} \quad \Rightarrow \quad f'(1) = -\frac{1}{2}$$

$$f''(x) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} \quad \Rightarrow \quad f''(1) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)$$

$$f'''(x) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2} \quad \Rightarrow \quad f'''(1) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)$$

$$f^{(4)}(x) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)x^{-9/2} \quad \Rightarrow \quad f^{(4)}(1) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)$$

Pattern: $f^{(n)}(1) = (-1)^n \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n}$ for $n \geq 1$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \boxed{1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!}(x-1)^n}$$

$$= 1 - \frac{1}{2}(x-1) + \frac{3}{8}(x-1)^2 - \frac{5}{16}(x-1)^3 + \frac{35}{128}(x-1)^4 + \cdots$$

$$\text{R.C.} = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)/2^n n!}{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)/2^{n+1}(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot 2^{n+1}(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1) \cdot 2^n \cdot n!}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)}{2n+1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n+2}{2n+1}\right| \overset{L'H}{=} \lim_{n\to\infty} \frac{2}{2} = \boxed{1}$$

5. (a) Find the function $f(x)$ whose Maclaurin series is $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$. (Hint: Multiply the series by $x$ and then differentiate.)

$$x f(x) = x \sum_{n=0}^{\infty} \frac{x^n}{n+1} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \Rightarrow \frac{d}{dx}\left(x f(x)\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\right)$$

$$= \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

$(-1 < x < 1)$

$$\therefore\ x f(x) = \int \frac{1}{1-x}\,dx = -\ln|1-x| + C$$

let $x=0$ to solve for $C$: $0 f(0) = 0 = -\ln 1 + C = 0 + C = C \Rightarrow \underline{C=0}$

$$\therefore\ x f(x) = -\ln|1-x|$$

$$\therefore\ \boxed{f(x) = \frac{-\ln|1-x|}{x}}$$

$(-1 < x < 1)$
$x \neq 0$

(Technically, this function is not defined at $x=0$. So to be completely correct, we should have $\boxed{f(x) = \begin{cases} \frac{-\ln|1-x|}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0. \end{cases}}$ )

(b) Use (a) to evaluate the sum $\sum_{n=0}^{\infty} \frac{1}{(n+1)2^n}$.

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)2^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{1}{2}\right)^n = f(1/2)$$

$$= \frac{-\ln|1-1/2|}{1/2}$$

$$= -2\ln 1/2$$

$$= 2\ln 2$$

$$= \boxed{\ln 4}$$