**Solutions to Final Exam Review Problems**

Math 5C, Winter 2007

1. Let $f(x) = \frac{1}{4+x}$.

   (a) Find the Maclaurin series for $f(x)$, and compute its radius of convergence.

   **Solution.** $f(x) = \dfrac{1}{4(1-(-x/4))} = \dfrac{1}{4}\sum_{n=0}^{\infty}(-x/4)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{4^{n+1}}x^n$. Since the infinite series is geometric, with ratio $-x/4$, it converges for $|-x/4| < 1$, and thus for $|x| < 4$. Thus, the radius of convergence is 4.

   (Note: it is also possible to use Taylor's formula, $c_n = f^{(n)}(0)/n!$, to determine the coefficients of the Maclaurin series and then use the ratio test to compute the radius of convergence.)

   (b) Find the Taylor series for $f(x)$ centered at $x = 1$, and compute its radius of convergence.

   **Solution.**

$$
\begin{aligned}
f(x) = \frac{1}{5+(x-1)} &= \frac{1}{5(1--(x-1)/5)} \\
&= \frac{1}{5}\sum_{n=0}^{\infty}\frac{-(x-1)}{5} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(x-1)^n.
\end{aligned}
$$

   Since the infinite series is geometric with ratio $-(x-1)/5$, it converges for $|-(x-1)/5| < 1$, and thus for $|x-1| < 5$. So the radius of convergence is 5.

   (Note: Again, we could have used Taylor's formula $c_n = f^{(n)}(1)/n!$, and the ratio test to find the radius of convergence.)

   (c) Find the Taylor series for $g(x) = \ln(4+x)$ centered at $x = 1$, and compute its radius of convergence.

   **Solution.** Since $g(x) = \int f(x)\,dx$, we integrate the Taylor series from part (b):

$$
g(x) = \int \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(x-1)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}(n+1)}(x-1)^{n+1} + C.
$$

   To find $C$, we plug in $x = 1$: $C = g(1) = \ln 5$. Thus, replacing $n+1$ by $k$, $g(x) = \ln 5 + \sum_{k=1}^{\infty}\dfrac{(-1)^{k-1}}{5^k k}(x-1)^k$. The radius of convergence is the same as the series we integrated, and thus equals 5.

2. For what values of $x$ does the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^x}$ converge? (Bonus: Does it converge uniformly on this entire set?)

**Solution.** We use the ratio test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)^x}{x^n/n^x} \right| &= \lim_{n\to\infty} \left| \frac{x}{(1+1/n)^x} \right| \\ &= |x/1^x| \\ &= |x| < 1 \end{aligned}$$

So it converges for $x$ in the interval $(-1, 1)$, but we still need to check whether it converges at the endpoints of this interval: $x = \pm 1$. If $x = 1$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges by the $p$-series test. If $x = -1$, the series becomes $\displaystyle\sum_{n=1}^{\infty} (-1)^n n$ which diverges by the nth term test, since $\lim_{n\to\infty} (-1)^n n \neq 0$. Thus the series converges only for $x$ in the interval $(-1, 1)$.

3. Show that the series of functions $\sum_{n=1}^{\infty} ne^{-nx}$ converges uniformly on $[1/2, \infty)$.

**Solution.** We use the M-test, where $u_n(x) = ne^{-nx}$. Since $x \geq 1/2$, and each $u_n(x)$ is a positive-valued decreasing function, $|u_n(x)| = u_n(x) \leq u_n(1/2) = ne^{-n/2}$. Thus, we let $M_n = ne^{-n/2}$. To show that the series converges uniformly, we must now check that the series $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} n/e^{n/2}$ converges. We can check this by the ratio test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{(n+1)/e^{(n+1)/2}}{n/e^{n/2}} \right| &= \lim_{n\to\infty} \frac{1+1/n}{e^{1/2}} \\ &= 1/\sqrt{e} < 1, \end{aligned}$$

meaning that the series converges by the ratio test.

4. Let $f(x) = \begin{cases} 1, & \text{if } -\pi/2 \leq x \leq \pi/2 \\ -1, & \text{if } -\pi \leq x < -\pi/2 \text{ or } \pi/2 < x \leq \pi \end{cases}$

   (a) Find the Fourier series for $f(x)$, and sketch its graph.

   **Solution.** Notice that $f(x)$ is an even function (it's graph is symmetric about the $y$-axis). Thus all the $b_n$'s will equal 0, and we only need to compute the $a_n$ coefficients. We have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\cos(nx)\ dx + \int_{-\pi/2}^{\pi/2} \cos(nx)\ dx + \int_{\pi/2}^{\pi} -\cos(nx)\ dx\right] \\
&= \frac{1}{\pi}\left(-\frac{\sin(nx)}{n}\Bigg]_{-\pi}^{-\pi/2} + \frac{\sin(nx)}{n}\Bigg]_{-\pi/2}^{\pi/2} + -\frac{\sin(nx)}{n}\Bigg]_{\pi/2}^{\pi}\right) \\
&= \frac{1}{\pi}\left(\frac{4\sin(n\pi/2)}{n}\right) \\
&= \begin{cases} 0, & n \text{ even} \\ 4(-1)^{(n-1)/2}/n\pi, & n \text{ odd} \end{cases}
\end{aligned}
$$

The above is correct for $n > 0$ only. When $n = 0$, the three integrals in the second line above evaluate to $-\pi/2, \pi, -\pi/2$, respectively. Thus $a_0 = 0$, and the Fourier series is

$$
\begin{aligned}
f(x) \sim \sum_{n=1}^{\infty} a_n \cos(nx) &= \sum_{n=1,\ n\ odd}^{\infty} \frac{4(-1)^{(n-1)/2}}{n\pi}\cos(nx) \\
&= \sum_{k=1}^{\infty} \frac{4(-1)^{k-1}}{(2k-1)\pi}\cos(2k-1)x \\
&= \frac{4}{\pi}\left(\cos x - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \frac{1}{7}\cos(7x) + \cdots\right).
\end{aligned}
$$

(Note: Any of the last 3 expressions would be a correct answer.) Its graph consists of horizontal segments at $y = 1$ over the intervals $((2k-1/2)\pi, (2k+1/2)\pi)$ and at $y = -1$ over the intervals $((2k+1/2)\pi, (2k+3/2)\pi)$ for each integer $k$, and it has points on the $x$-axis at the endpoints of each of these intervals.

(b) Use part (a) (or other methods) to find the Fourier series for

$$
g(x) = \begin{cases} -\pi - x, & \text{if } -\pi \le x < -\pi/2 \\ x, & \text{if } -\pi/2 \le x \le \pi/2 \\ \pi - x & \text{if } \pi/2 < x \le \pi \end{cases}
$$

**Solution.** Notice that $g'(x) = f(x)$. Thus we can obtain the Fourier series for $g(x)$ by integrating our answer to part (a).

$$
\begin{aligned}
g(x) &\sim \int \sum_{k=1}^{\infty} \frac{4(-1)^{k-1}}{(2k-1)\pi}\cos(2k-1)x\ dx \\
&= C + \sum_{k=1}^{\infty} \frac{4(-1)^{k-1}}{(2k-1)^2\pi}\sin(2k-1)x \\
&= C + \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin(3x) + \frac{1}{5^2}\sin(5x) - \frac{1}{7^2}\sin(7x) + \cdots\right).
\end{aligned}
$$

Since $g(x)$ is an odd function, all the $a_n$ coefficients, including $a_0$, equal 0. Thus $C = a_0/2 = 0$, and removing the $C$'s from the above yields the correct Fourier series.

(c) Show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$. (You should practice using one of the Fourier series above, or else one from lecture, rather than a power series.)

**Solution.** Plug $x = 0$ into the Fourier series from (a) to get $f(0) = \frac{4}{\pi}(1 - 1/3 + 1/5 - 1/7 + \cdots)$. Since $f(0) = 1$, the sum $1 - 1/3 + 1/5 - 1/7 + \cdots$ converges to $\pi/4$.

5. Find the Fourier series of $f(x) = e^x$. (Suggestion: use the complex form of the Fourier series.)

**Solution.** The complex form of the Fourier series is $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$, where the coefficients $c_n$ are given by the formula $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$. Here,

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,dx \\
&= \left.\frac{e^{(1-in)x}}{2\pi(1-in)}\right]_{-\pi}^{\pi} \\
&= \frac{e^{(1-in)\pi} - e^{-(1-in)\pi}}{2\pi(1-in)} \\
&= \frac{e^{\pi}(\cos(-n\pi) + i\sin(-n\pi)) - e^{-\pi}(\cos(n\pi) + i\sin(n\pi))}{2\pi(1-in)} \\
&= \frac{(-1)^n e^{\pi} - (-1)^n e^{-\pi}}{2\pi(1-in)} \\
&= \frac{(-1)^n(e^{\pi} - e^{-\pi})(1+in)}{2\pi(n^2+1)} \\
&= \frac{(-1)^n(e^{\pi} - e^{-\pi})}{2\pi(n^2+1)} + i\frac{(-1)^n(e^{\pi} - e^{-\pi})n}{2\pi(n^2+1)}
\end{aligned}$$

Thus

$$e^x \sim \sum_{n=-\infty}^{\infty}\left(\frac{(-1)^n(e^{\pi} - e^{-\pi})}{2\pi(n^2+1)} + i\frac{(-1)^n(e^{\pi} - e^{-\pi})n}{2\pi(n^2+1)}\right)e^{inx}.$$

Since $c_n = (a_n - ib_n)/2$ for $n > 0$, and $c_0 = a_0/2$, we see that $a_n$ equals twice the real part of $c_n$ for each $n \geq 0$, and $b_n$ equals (-1) times twice the imaginary part of $c_n$ for $n > 0$. Thus, in terms of sines and cosines,

$$e^x \sim \frac{e^{\pi} - e^{-\pi}}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2+1}\cos(nx) + \frac{(-1)^{n+1}n}{n^2+1}\sin(nx)\right)\right).$$

6. Find the solution of the wave equation $\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^u}{\partial x^2} = 0$ with initial displacement given by $f(x) = 0$, and initial velocity given by $g(x) = \sin^2 x$. (Hint: use a half-angle formula!) Sketch the solution when $t = \pi/2$.

**Solution.** Due to an error on my part, the Hint is useless. We have to solve the wave equation with $a = 2$, $f(x) = 0$, and $g(x) = \sin^2 x$. The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)[\alpha_n \sin(2nt) + \beta_n \cos(2nt)]$$

where $2n\alpha_n$ are the Fourier sine coefficients of $g(x) = \sin^2 x$ and $\beta_n$ are the Fourier sine coefficients of $f(x) = 0$. Thus all the $\beta_n$ are 0, and the $\alpha_n$'s are computed as in lecture (see the lecture notes pp. 66-7):

$$\alpha_n = \frac{1}{2n}\left(\frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin(nx)\, dx\right) = \begin{cases} 0, & n \text{ even} \\ \frac{-4}{n^2\pi(n^2-4)}, & n \text{ odd} \end{cases}$$

Hence, the solution as a Fourier series is

$$\begin{aligned} u(x,t) &= \sum_{n=1,\ odd}^{\infty} \frac{-4}{n^2\pi(n^2-4)} \sin(nx)\sin(2nt) \\ &= \sum_{k=1}^{\infty} \frac{-4}{(2k-1)^2\pi((2k-1)^2-4)} \sin((2k-1)x)\sin(2(2k-1)t). \end{aligned}$$

When $t = \pi/2$, $\sin(2nt) = \sin(n\pi) = 0$. Thus $u(x, \pi/2) = 0$, and the string is in equilibrium position along the $x$-axis.