Math 5C, Solutions to Midterm 1 Review Problems

Winter 2007

1. Compute $\int_C yz\ dx + 2x\ dy - y\ dz$ where C is the straight line path from $(1, 2, 1)$ to $(-1, 3, 0)$.

   Solution. The straight line path between the two points has parametric equations $(x, y, z) = (-2t + 1, t + 2, -t + 1)$. Thus, using the definition of line integrals, the integral becomes

$$\begin{aligned}\int_0^1 (t+2)(1-t)(-2) + 2(1-2t) - (t+2)(-1)\ dt &= \int_0^1 2t^2 - t\ dt \\ &= 2t^3/3 - t^2/2]_0^1 = 1/6.\end{aligned}$$

2. Find the surface area of the surface S, which is parametrized by

$$\phi(u, v) = \begin{cases} x(u, v) = u - v \\ y(u, v) = u + v \\ z(u, v) = uv \end{cases}$$

   for all (u, v) with $u^2 + v^2 \leq 1$.

   Solution. We use the formula $S.A. = \int\int \sqrt{EG - F^2}\ du\ dv$. Here, $E = x_u^2 + y_u^2 + z_u^2 = 2 + v^2$, $F = x_u x_v + y_u y_v + z_u z_v = 1 - 1 + vu = uv$, and $G = x_v^2 + y_v^2 + z_v^2 = 2 + u^2$. Thus $EG - F^2 = 4 + 2(u^2 + v^2)$, and

$$\begin{aligned} S.A. &= \int\int_{u^2+v^2\leq 1} \sqrt{4 + 2(u^2 + v^2)}\ du\ dv \\ &= \int_0^{2\pi}\int_0^1 \sqrt{4 + 2r^2}\ r\ dr\ d\theta \\ &= \frac{1}{4}\int_0^{2\pi} \frac{2}{3}(4 + 2r^2)^{3/2}]_0^1\ d\theta \\ &= \frac{\pi}{3}(6^{3/2} - 4^{3/2}) = \frac{\pi}{3}(6\sqrt{6} - 8)\end{aligned}$$

3. Let S be the top half of the unit sphere (i.e., S is given by $x^2 + y^2 + z^2 = 1$ and $z \geq 0$), oriented by the outer normal. Integrate

$$\int\int_S x\ dy\ dz + y\ dz\ dx + z^2\ dx\ dy.$$

   Solution. S has parametrization $x = \cos\theta\sin\phi$, $y = \sin\theta\sin\phi$ and $z = \cos\phi$ for $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$. Notice that this parametrization induces the inner normal on the sphere, and so we need to multiply the integral by -1 when we write it in terms of the parameters θ and ϕ.

We now compute the Jacobians we need (these are also the components of the normal vector).

$$dy\ dz = \begin{vmatrix} \cos\theta\sin\phi & \sin\theta\cos\phi \\ 0 & -\sin\phi \end{vmatrix} d\theta\ d\phi = -\cos\theta\sin^2\phi\ d\theta\ d\phi,$$

$$dz\ dx = \begin{vmatrix} 0 & -\sin\phi \\ -\sin\theta\sin\phi & \cos\theta\cos\phi \end{vmatrix} d\theta\ d\phi = -\sin\theta\sin^2\phi\ d\theta\ d\phi,$$

$$dx\ dy = \begin{vmatrix} -\sin\theta\sin\phi & \cos\theta\cos\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \end{vmatrix} d\theta\ d\phi = -\sin\phi\cos\phi\ d\theta\ d\phi,$$

Thus

$$\begin{aligned}
\int\int_S x\ dy\ dz + \cdots &= -\int_0^{\pi/2}\int_0^{2\pi} -\cos^2\theta\sin^3\phi - \sin^2\theta\sin^3\phi - \sin\phi\cos^3\phi\ d\theta\ d\phi \\
&= 2\pi\int_0^{\pi/2}(\sin^2\phi + \cos^3\phi)\sin\phi\ d\phi \\
&= 2\pi\int_0^{\pi/2}(1-\cos^2\phi + \cos^3\phi)\sin\phi\ d\phi \\
&= -2\pi(\cos\phi - \frac{1}{3}\cos^3\phi + \frac{1}{4}\cos^4\phi)]_0^{\pi/2} = \frac{11\pi}{6}.
\end{aligned}$$

4. Let S be the surface given by $z = xy^2 - 3x^2$ with upper normal $\mathbf{n}$, over the square with vertices $(\pm 1, \pm 1)$ in the xy-plane. If $\mathbf{w} = (z+3x^2)\mathbf{i} + yz\mathbf{j} + y^2\mathbf{k}$, calculate

$$\int\int_S \mathbf{w}\cdot\mathbf{n}\ d\sigma.$$

Solution.

$$\begin{aligned}
\int\int_S \mathbf{w}\cdot\mathbf{n}\ d\sigma &= \int_{-1}^1\int_{-1}^1 (-w_x\frac{\partial z}{\partial x} - w_y\frac{\partial z}{\partial y} + w_z)\ dx\ dy \\
&= \int_{-1}^1\int_{-1}^1 -(xy^2-3x^2+3x^2)(y^2-6x) - y(xy^2-3x^2)(2xy) + y^2\ dx\ dy \\
&= \int_{-1}^1\int_{-1}^1 6x^2y^2 - xy^4 - 2x^2y^4 + 6x^3y^2 + y^2\ dx\ dy \\
&= \int_{-1}^1 6y^2 - 4y^4/3\ dy = \frac{52}{15}.
\end{aligned}$$

5. Let S be the unit sphere, $x^2 + y^2 + z^2 = 1$, oriented outward, and let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = xy^2\ \mathbf{i} - xz^2\ \mathbf{j} + x^2 z\ \mathbf{k}$. Use the Divergence Theorem to compute $\int\int_S \mathbf{F} \cdot \mathbf{n}\ d\sigma$.

Solution.

$$\begin{aligned}
\int\int_S \mathbf{F} \cdot \mathbf{n}\ d\sigma &= \int\int\int_R \text{div}(\mathbf{F})\ dxdydz \\
&= \int\int\int_R y^2 + x^2\ dxdydz \\
&= \int_0^\pi \int_0^{2\pi} \int_0^1 (\rho^2 \sin^2\phi)\rho^2 \sin\phi\ d\rho d\theta d\phi \\
&= \int_0^\pi \int_0^{2\pi} \frac{1}{5} \sin^3\phi\ d\theta d\phi \\
&= \int_0^\pi \frac{2\pi}{5}(1 - \cos^2\phi)\sin\phi\ d\phi \\
&= -\frac{2\pi}{5}(\phi - \frac{1}{3}\cos^3\phi)]_0^\pi \\
&= -\frac{2\pi}{5}(\pi + 1/3 - 0 - -1/3) = -\frac{2\pi^2}{5} - \frac{4\pi}{15}
\end{aligned}$$

6. Let S be the cone $x^2 = y^2 + z^2,\ \ 0 \le x \le 2$, oriented inward (so the normal vectors point toward the x-axis). Use Stokes' Theorem to calculate $\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\ d\sigma$, where $\mathbf{F}(x, y, z) = x^2\mathbf{i} - z\mathbf{j} + (y^2 - z)\mathbf{k}$.

Solution. Recall $\nabla \times \mathbf{F} = \text{curl}(\mathbf{F})$, so by Stokes' Theorem, the surface integral reduces to the line integral $\oint_C F_T\ ds$ where C is the boundary of S. Now S is a cone with vertex at the origin, and so its boundary is a circle lying in the plane $x = 2$. By the right-hand rule, since the normal should be roughly in the direction of the x-axis, the circle C should be traversed in the counterclockwise direction of the yz-plane. Thus C has parametric equations $(x, y, z) = (2, \cos t, \sin t)$ for $0 \le t \le 2\pi$, and

$$\begin{aligned}
\int_C F_T\ ds &= \int_C x^2\ dx - z\ dy + (y^2 - z)\ dz \\
&= \int_0^{2\pi} 4(0) - \sin t(-\sin t) + (\cos^2 t - \sin t)\cos t\ dt \\
&= \int_0^{2\pi} \sin^2 t + (cos^2 t - \sin t)\cos t\ dt \\
&= \int_0^{2\pi} \frac{1}{2}(1 - \cos(2t)) + (1 - \sin t - \sin^2 t)\cos t\ dt \\
&= \frac{1}{2}(t - \frac{1}{2}\sin(2t)) + \sin t - \frac{1}{2}\sin^2 t - \frac{1}{3}\sin^3 t]_0^{2\pi} = \pi.
\end{aligned}$$

7. Let C be the curve given by $x = \sin t,\ y = \cos t,\ z = \cos(2t)$ for $0 \le t \le 2\pi$. Use Stokes' Theorem to evaluate

$$\oint_C xz\ dx + y^2\ dy + z^2\ dz.$$

Solution. Since $z = \cos(2t) = \cos^2 t - \sin^2 t = y^2 - x^2$, the curve C lies on the surface $z = y^2 - x^2$ (the surfaces $z = 2y^2 - 1$ or $z = 1 - 2x^2$ would also work). Since $x = \sin t,\ y = \cos t$ traces out the unit circle, the interior of C lies above the unit disk in the xy-plane. Thus C is the boundary of the surface S given by the graph of $z = y^2 - x^2$ for $x^2 + y^2 \le 1$. Furthermore, since C is traversed in the clockwise direction, S must be given the lower normal according to the right-hand rule (thus we multiply by -1). Now, using Stokes' Theorem, we have

$$\begin{aligned}
\oint_C xz\ dx + y^2\ dy + z^2\ dz &= \int\int_S \operatorname{curl}(xz\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot \mathbf{n}\ d\sigma \\
&= -\int\int_S x\ dz\ dx \\
&= -\int\int_{x^2+y^2\le 1} -x(2y)\ dx\ dy \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2xy\ dy\ dx \\
&= \int_{-1}^{1} 2x(1-x^2)\ dx = x^2 - x^4/2]_{-1}^{1} = 0.
\end{aligned}$$

8. Show that the integral

$$\int_{(-1,1,3)}^{(\pi/2,0,1)} z^2\cos(x+y^2)\ dx + 2yz^2\cos(x+y^2)\ dy + 2z\sin(x+y^2)\ dz$$

is independent of path and evaluate it.

Solution. To show that the integral is path independent, it suffices to find a function $F(x, y, z)$ such that the integrand equals dF. To get F, integrate the dx term with respect to x to get $F = \int z^2\cos(x+y^2)\ dx = z^2\sin(x+y^2) + C(y, z)$. If we now differentiate this function with respect to y and z (separately), we get the other two terms of the integrand when we let $C(y, z) = 0$. Thus the integral becomes

$$\begin{aligned}
\int_{(-1,1,3)}^{(\pi/2,0,1)} d(z^2\sin(x+y^2)) &= z^2\sin(x+y^2)]_{(-1,1,3)}^{(\pi/2,0,1)} \\
&= \sin(\pi/2) - 9\sin 0 = 1.
\end{aligned}$$

9. Let $\mathbf{u}$ be the vector field

$$\mathbf{u}(x, y, z) = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j} + z^2\mathbf{k}$$

on $\mathbb{R}^3$ minus the z-axis.

(a) Show that $\text{curl}(\mathbf{u}) = \mathbf{0}$ on this domain.

Solution.

$$\text{curl}(\mathbf{u}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{x^2+y^2} & \frac{-x}{x^2+y^2} & z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (\frac{x^2 - y^2}{(x^2 + y^2)^2} - \frac{x^2 - y^2}{(x^2 + y^2)^2})\mathbf{k} = \mathbf{0}.$$

(b) Show that $\mathbf{u}$ is not the gradient vector field of any function F on this domain. (Hint: Find a closed curve C with $\int_C u_T \, ds \neq 0$.)

Solution. Let C be the unit circle in the xy-plane: $x = \cos t,\ y = \sin t,\ z = 0$ for $0 \leq t \leq 2\pi$. Then

$$\begin{aligned} \oint_C u_T \, ds &= \int_0^{2\pi} (\frac{\sin t}{\cos^2 t + \sin^2 t}(-\sin t) - \frac{\cos t}{\cos^2 t + \sin^2 t}(\cos t) + 0) \, dt \\ &= \int_0^{2\pi} -(\sin^2 t + \cos^2 t) \, dt \\ &= -2\pi \neq 0. \end{aligned}$$

Since this integral is not zero, we know that the integral $\int u_T \, ds$ is not path-independent in the given domain, and hence $\mathbf{u}$ is not a gradient vector field. (You could also prove this more directly by trying to solve for $F(x, y, z)$ with $\nabla F = \mathbf{u}$, and showing that no solutions exist.)