

① Math 5C - Solutions to Midterm 2 Review Problems

1. a) Geometric: $a = -4, r = -1/4$

$$a_n = -4(-1/4)^n \quad \text{for } n \geq 0$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{-4(-1)^n}{4^n} = \lim_{n \rightarrow \infty} \frac{(-1)^{n+1}}{4^{n-1}} = \boxed{0}$$

$$b) a_n = \frac{2n-1}{2n+1}, n \geq 1$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2n-1}{2n+1} \stackrel{LH}{=} \lim_{n \rightarrow \infty} \frac{2}{2} = \boxed{1}$$

$$c) a_n = \begin{cases} (-1)^{n+1}(n^2-1), & \text{if } n \geq 2 \\ 1, & \text{if } n=1 \end{cases} \quad \text{for } n \geq 1$$

(there is a typo in the problem: a_1 should be 0, not 1)
then $a_n = (-1)^{n+1}(n^2-1)$ for $n \geq 1$.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (-1)^{n+1}(n^2-1) = \boxed{\text{D.N.E.}}$$

since it oscillates between large negative values & large positive values.

$$d) a_n = \frac{2n+1}{2^n} \quad \text{for } n \geq 0$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2n+1}{2^n} \stackrel{LH}{=} \lim_{n \rightarrow \infty} \frac{2}{2^n \cdot \ln 2} = \boxed{0}$$

2. a) $\sum_{n=1}^{\infty} 2^{1-2n} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \dots$
= geometric series w/ $a = 1/2, r = 1/4$.

$$\text{sum} = \frac{a}{1-r} = \frac{1/2}{1-1/4} = \frac{1/2}{3/4} = \boxed{\frac{2}{3}}$$

$$b) \sum_{n=1}^{\infty} \left(\frac{\sqrt{2n+1}}{n} - \frac{\sqrt{2n+3}}{n+1} \right) = \left(\frac{\sqrt{3}}{1} - \frac{\sqrt{5}}{2} \right) + \left(\frac{\sqrt{5}}{2} - \frac{\sqrt{7}}{3} \right) + \left(\frac{\sqrt{7}}{3} - \frac{\sqrt{9}}{4} \right) + \dots$$

N^{th} partial sum = $\sqrt{3} - \frac{\sqrt{2N+3}}{N+1}$ (telescoping series)

$$\text{sum} = \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \left(\sqrt{3} - \frac{\sqrt{2N+3}}{N+1} \right) = \boxed{\sqrt{3}}$$

②

3 a) $\sqrt{\frac{n+1}{3n^2}} \geq \sqrt{\frac{n}{3n^2}} = \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{n}}\right)$ for all $n \geq 1$.

and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3}} \left(\frac{1}{\sqrt{n}}\right)$ diverges since it is a p-series w/ $p = 1/2 < 1$.

\therefore By comparison test, $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{3n^2}}$ diverges.

b) $a_n = \frac{1}{n \ln n}$ for $n \geq 2$ is positive and decreasing.

\therefore By integral test, since $\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x \ln x} dx$

$= \lim_{b \rightarrow \infty} \ln |\ln x| \Big|_2^b = \lim_{b \rightarrow \infty} (\ln(\ln b) - \ln(\ln 2)) = \infty$ Diverges,

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ Diverges too.

c) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ is an alternating series w/ $b_n = \frac{\ln n}{n}$.
 $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} \stackrel{\text{L'H}}{=} \lim_{n \rightarrow \infty} \frac{1/n}{1} = 0 \checkmark$

$\frac{\ln n}{n}$ is decreasing for $n \geq 3$. \checkmark

$f(x) = \ln x / x \Rightarrow f'(x) = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0$
 when $x > e$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ converges by the Alternating Series Test.

d) Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^{n+1} / (2 \cdot 4 \cdots 2n)(2n+2)}{n^n / (2 \cdot 4 \cdots 2n)} \right|$

$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(n+1)^n}{2n+2} \cdot \frac{1}{n^n} \right| = \frac{1}{2} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \frac{1}{2} e > 1$

So it Diverges by the Ratio test.

4a) Ratio test $\lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1} / 4^{n+1}}{(x-3)^n / 4^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x-3}{4} \right| = \frac{|x-3|}{4} < 1$

$\Rightarrow |x-3| < 4 \Rightarrow \boxed{R=4}$.

end points: $x = 3-4 = -1$ & $x = 3+4 = 7$.

if $x = -1$, we get $\sum_{n=0}^{\infty} \frac{(-4)^n}{4^n} = \sum_{n=0}^{\infty} (-1)^n$ Diverges

Since $\lim_{n \rightarrow \infty} (-1)^n \neq 0$.

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If $x=7$, we get $\sum_{n=0}^{\infty} \frac{4^n}{4^n} = \sum_{n=0}^{\infty} 1$ Diverges

Since $\lim_{n \rightarrow \infty} 1 = 1 \neq 0$.

So I.C. = (-1, 7)

b) Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}/n+2}{x^{2n}/n+1} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^2(n+1)}{n+2} \right|$
 $= |x^2| \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \right| = |x^2| < 1$

Thus we need $|x| < 1$, \therefore R = 1.

AT endpoints $x = \pm 1$:

If $x = -1$, we get $\sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ Diverges b/c it's the Harmonic series

If $x = 1$, we get $\sum_{n=0}^{\infty} \frac{1}{n+1}$ Diverges (as above)

\therefore I.C. = (-1, 1)

5. The series $\sum_{n=0}^{\infty} c_n(x-1)^n$ is centered at $x=1$.

Since it converges at $x=2$, R.C. $\geq |2-1| = 1$.

Since it Diverges at $x=-3$, R.C. $\leq |-3-1| = 4$.

a) at $x=1$, it converges to c_0

b) at $x=0$, we can't tell since $|0-1| = 1$

may equal the R.C. (eg. I.C. could be $(0, 2]$ or $[0, 2]$)

c) at $x=1/2$, $|x-1| = |1/2-1| = 1/2 < 1 \leq$ R.C.

So $x=1/2$ is in the interval of convergence

$\&$ it converges.

d) $x=3$, we can't tell since I.C. could be $[0, 2]$ or $[-1, 3]$.

e) $x=5$, we can't tell since I.C. could be $(-3, 5]$ or $(-3, 5)$

f) $x=6$, ~~we can't~~ $|x-1| = |6-1| = 5 > 4 \geq$ R.C.

\therefore $x=6$ is not in the I.C. $\&$ it Diverges.

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6. Notice $f(x) = x^2 \left(\frac{1}{(1+x)^2} \right)$ and

$$\frac{1}{(1+x)^2} = -\frac{d}{dx} \left(\frac{1}{1+x} \right)$$

$$\text{Since } \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n \quad (\text{w/ } R=1)$$

$$\begin{aligned} \frac{1}{(1+x)^2} &= -\frac{d}{dx} \left(\frac{1}{1+x} \right) = -\frac{d}{dx} \left(\sum_{n=0}^{\infty} (-x)^n \right) \\ &= -\sum_{n=0}^{\infty} \frac{d}{dx} [(-1)^n x^n] \end{aligned}$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} n x^{n-1}$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \quad (R=1)$$

$$\text{Thus } f(x) = x^2 \left(\frac{1}{(1+x)^2} \right)$$

$$= x^2 \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^{n+2} = \boxed{\sum_{n=2}^{\infty} (-1)^n (n-1) x^n}$$

$R=1$, so to find I.C. we need to check convergence at the endpoints $x = \pm 1$.

$$\text{at } x = -1, \text{ we get } \sum_{n=2}^{\infty} (-1)^{2n} (n-1) = \sum_{n=2}^{\infty} n-1 \quad \boxed{\text{Diverges}}$$

$$\text{Since } \lim_{n \rightarrow \infty} (n-1) \neq 0.$$

$$\text{at } x = 1, \text{ we get } \sum_{n=2}^{\infty} (-1)^n (n-1) = \boxed{\text{Diverges}}$$

$$\text{Since } \lim_{n \rightarrow \infty} (-1)^n (n-1) \neq 0.$$

$$\therefore \boxed{\text{I.C.} = (-1, 1)}.$$

$$7. \text{ Let } f(x) = \sum_{n=1}^{\infty} n x^n. \text{ Then } \sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n \left(\frac{1}{2} \right)^n = f\left(\frac{1}{2} \right)$$

so we must find $f(x)$.

$$\text{Notice } \frac{1}{(1-x)^2} = \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\therefore \frac{x}{(1-x)^2} = \left(\sum_{n=1}^{\infty} n x^{n-1} \right) x = \sum_{n=1}^{\infty} n x^n = f(x). \quad \therefore f\left(\frac{1}{2} \right) = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = \boxed{2}$$