Homework 3 Solutions

p319 #1

(a) $\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \iiint_R \operatorname{div}(x,y,z)\,dx\,dy\,dz$

$$= \iiint_R 3\,dx\,dy\,dz = 3\cdot(\text{Volume of } R) = \boxed{4\pi}$$

(b) $\iint_S x^2\,dy\,dz + y^2\,dz\,dx + z^2\,dx\,dy = \iiint_R \operatorname{div}(x^2, y^2, z^2)\,dx\,dy\,dz$

$$= \iiint_R (2x+2y+2z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 (2x+2y+2z)\,dx\,dy\,dz = 1+1+1 = \boxed{3}$$

(c) $\iint_S e^y\cos z\,dy\,dz + e^x\sin z\,dz\,dx + e^x\cos y\,dx\,dy$

$$= \iiint_R \operatorname{div}(e^y\cos z, e^x\sin z, e^x\cos y)\,dx\,dy\,dz$$

$$= \iiint_R 0\cdot dx\,dy\,dz = \boxed{0}$$

(d) $\iint_S \nabla F\cdot\vec{n}\,d\sigma = \iint_S \nabla(x^2+y^2+z^2)\cdot\vec{n}\,d\sigma = \iint_S (2x, 2y, 2z)\cdot\vec{n}\,d\sigma$

$$= \iiint_R 6\,dx\,dy\,dz = \boxed{6\cdot(\text{Volume of } R)}$$

(e) $\iint_S \nabla F\cdot\vec{n}\,d\sigma = \iint_S \nabla(2x^2-y^2-z^2)\cdot\vec{n}\,d\sigma = \iint_S (4x, -2y, -2z)\cdot\vec{n}\,d\sigma$

$$= \iiint_R 0\cdot dx\,dy\,dz = \boxed{0}$$

p319 #2

(a) $V = \iiint_R dx\,dy\,dz$

$$= \iiint_R \operatorname{div}(x,0,0)\,dx\,dy\,dz$$

$$= \iint_S x\,dy\,dz + 0\,dz\,dx + 0\,dx\,dy$$

$$= \iint_S x\,dy\,dz$$

By considering the vector field $(0,y,0)$, we obtain $V = \iint_S y\,dz\,dx$.

By considering the vector field $(0,0,z)$, we obtain $V = \iint_S z\,dx\,dy$.

So $3V = \iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$.

(c) Let $\vec{v} = (f(x,y,z), g(x,y,z), h(x,y,z))$ be a (smooth) vector field. So $\operatorname{curl}\vec{v} = (h_y - g_z, f_z - h_x, g_x - f_y)$

and $\operatorname{div}(\operatorname{curl}\vec{v}) = (h_y - g_z)_x + (f_z - h_x)_y + (g_x - f_y)_z$

$$= h_{yx} - g_{zx} + f_{zy} - h_{xy} + g_{xz} - f_{yz}$$

$$= 0$$ (by equality of mixed partial derivatives)

Thus $\iint_S \operatorname{curl}\vec{v}\cdot\vec{n}\,d\sigma = \iiint_R \operatorname{div}(\operatorname{curl}\vec{v})\,dx\,dy\,dz$

$$= \iiint_R 0\cdot dx\,dy\,dz = \boxed{0}$$

p330 #1

ⓐ $\int_C u_T \, ds = \iint_S \text{curl}\, \vec{u} \cdot n \, d\sigma$

where $S = \{(x,y,z) \in \mathbb{R}^3 \,/\, x^2+y^2 \leq 1 \text{ and } z=2\}$

and $\vec{n} = (0,0,1)$

So $\int_C u_T \, ds = \iint_S (\text{curl}\,\vec{u}) \cdot \vec{n} \, d\sigma$

$= \iint_S (0, 0, 6) \cdot \vec{n} \, d\sigma$

$= \iint_S 6 \, dx\,dy$

$= \boxed{6\pi}$

ⓑ $\int_C 2xy^2z\,dx + 2x^2yz\,dy + (x^2y^2 - 2z)\,dz$

$= \iint_S \text{curl}\,(2xy^2z, 2x^2yz, x^2y^2-2z) \cdot \vec{n} \, d\sigma$

where S is a surface with boundary C.

Here we have $\text{curl}\,(2xy^2z, 2x^2yz, x^2y^2-2z) = \vec{0}$

Thus $\int_C 2xy^2z\,dx + 2x^2yz\,dy + (x^2y^2-2z)\,dz = \boxed{0}$

p330 #3

C = simple closed curve in the plane $ax+by+cz=d$

S = surface in the plane $ax+by+cz=d$ having boundary C

$\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ unit normal. Thus $a^2+b^2+c^2=1$.

$$\frac{1}{2}\int_C (bz-cy)\,dx + (cx-az)\,dy + (ay-bx)\,dz$$

$$= \frac{1}{2}\iint_S \operatorname{curl}(bz-cy,\ cx-az,\ ay-bx)\cdot\vec{n}\,d\sigma$$

$$= \frac{1}{2}\iint_S (a+a,\ b+b,\ c+c)\cdot\vec{n}\,d\sigma$$

$$= \iint_S (a,b,c)\cdot\vec{n}\,d\sigma$$

$$= \iint_S (a,b,c)\cdot(a,b,c)\,d\sigma$$

$$= \iint_S (a^2+b^2+c^2)\,d\sigma$$

$$= \iint_S d\sigma$$

= surface area of the surface S

When C is in the xy-plane, we have $a=b=0$ and $c=1$ and the equation is

$$\frac{1}{2}\int_C -y\,dx + x\,dy = \text{area of enclosed region}$$