Homework 4 Solutions

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(2) (a) $f(x,y,z) = xyz$

$$df = yz\,dx + xz\,dy + xy\,dz$$

$$\int_{(1,1,2)}^{(3,5,0)} df = f(3,5,0) - f(1,1,2) = 0 - 2 = \boxed{-2}$$

(b) $g(x,y,z) = x\sin(yz)$

$$dg = \sin(yz)dx + xz\cos(yz) + xy\cos(yz)dz$$

$$\int_{(1,0,0)}^{(1,0,2\pi)} dg = g(1,0,2\pi) - g(1,0,0) = 0 - 0 = \boxed{0}$$

(4)

$$\operatorname{curl}\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z\right) = \left(0-0, 0-0, \frac{(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} + \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}\right)$$

$$= (0,0,0)$$

Let $C(t) = (2\cos t, 2\sin t, 0)$ for $0 \le t \le 2\pi$

then $$\oint_C u_T\,ds = \int_0^{2\pi}\left(\frac{-2\sin t}{4}\cdot(-2\sin t) + \frac{2\cos t}{4}(2\cos t) + 0\right)dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= 2\pi$$

The possible values for $\oint_C u_T\,ds$ are $2n\pi$ where $n \in \mathbb{Z}$ (n is the number of times C wraps around the z-axis in the clockwise direction)

Math 5C, Homework Problems from Lecture

Due: 7 Feb. 2007

These problems complete the proof of Gauss's Law from lecture (1/30). You may print out this page and write your solutions on it.

1. Consider the vector field
$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}},$$
which is differentiable on the domain $D = \mathbb{R}^3 - \{(0,0,0)\}$.

   (a) Show that $\operatorname{div}(\mathbf{F}) = 0$ on D.

   (b) If S is a sphere of radius r centered at the origin and with outer normal $\mathbf{n}$, show that
$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 4\pi.$$
(Notice that on S, $\mathbf{F}$ simplifies to $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/r^3$.)

(a)
$$\operatorname{div} F = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{1}{(x^2+y^2+z^2)^3}\Big[(x^2+y^2+z^2)^{3/2} - x\cdot\tfrac{3}{2}\sqrt{x^2+y^2+z^2}\cdot 2x + (x^2+y^2+z^2)^{3/2} - y\cdot\tfrac{3}{2}\sqrt{x^2+y^2+z^2}\cdot 2y + (x^2+y^2+z^2)^{3/2} - z\cdot\tfrac{3}{2}\sqrt{x^2+y^2+z^2}\cdot 2z\Big]$$

$$= \frac{1}{(x^2+y^2+z^2)}\left[3(x^2+y^2+z^2)^{3/2} - (3x^2+3y^2+3z^2)\sqrt{x^2+y^2+z^2}\right]$$

$$= \boxed{0}$$

(b) The outer normal vector is

$$\hat{n} = \frac{(x, y, z)}{\sqrt{x^2+y^2+z^2}} = \frac{1}{r}(x, y, z)$$

On S, we have $\vec{F} = \frac{1}{r^3}(x, y, z)$

Thus $\vec{F}\cdot\hat{n} = \frac{1}{r^4}(x^2+y^2+z^2) = \frac{1}{r^2}$ on S.

Therefore, use $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$.

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_S \frac{1}{r^2}\,d\sigma$$

$$= \int_0^{\pi}\int_0^{2\pi}\left(\frac{1}{r^2}\right) r^2 \sin\phi\, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \sin\phi\, d\theta\, d\phi$$

$$= 2\pi\int_0^{\pi} \sin\phi\, d\phi$$

$$= \boxed{4\pi}$$