Homework 5 Solutions

① For each of the following sequences, find a formula for the nth term a_n in terms of n, decide whether the sequence converges or diverges, and compute $\lim_{n\to\infty} a_n$ if it converges

(a) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\ldots$

$$a_n = \frac{1}{2n+1} \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n+1} = 0$$

(b) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}$

$$a_n = \frac{(-1)^{n+1}}{n} \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0$$

(c) $\frac{5}{2}, \frac{1}{2}, \frac{1}{10}, \frac{1}{50}, \ldots$

$$a_n = \frac{1}{(2/5)\,5^n} = \frac{1}{2\cdot 5^{n-1}} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{2\cdot 5^{n-1}} = 0$$

(d) $0, \frac{3}{2}, \frac{3}{4}, \frac{9}{8}, \frac{15}{16}, \frac{33}{32}$

$$a_n = \frac{2^n-(-1)^n}{2^n} \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n-(-1)^n}{2^n} = 1$$

(2)

(a) $\sum_{n=1}^{\infty} 3^{-n} = 1/3 + 1/9 + 1/27 + 1/81 + \frac{1}{243} + \cdots$

$S_1 = 1/3$

$S_2 = 1/3 + 1/9 = 4/9$

$S_3 = 1/3 + 1/9 + 1/27 = 13/27$

$S_4 = 1/3 + 1/9 + 1/27 + \frac{1}{81} = 40/81$

$S_5 = 1/3 + 1/9 + 1/27 + 1/81 + 1/243 = 121/243$

The series appears to converge to $\boxed{1/2}$.

(b) $\sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1} = 4 - 4/3 + 4/5 - 4/7 + 4/9 - \cdots$

$S_0 = 4$

$S_1 = 4 - 4/3 = 8/3$

$S_2 = 4 - 4/3 + 4/5 = 52/15$

$S_3 = 4 - 4/3 + 4/5 - 4/7 = 304/105$

$S_4 = 4 - 4/3 + 4/5 - 4/7 + 4/9 = \frac{1052}{315}$

The series appears to converge to $\boxed{3}$.

(c) $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots$

$S_0 = 1$

$S_1 = 1 + 1 = 2$

$S_2 = 1 + 1 + \frac{1}{2} = \frac{5}{2}$

$S_3 = 1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{16}{6} = \frac{8}{3}$

$S_4 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = \frac{65}{24} \approx 2.7083$

$S_5 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{326}{120} = \frac{163}{60} \approx 2.7167$

The series appears to converge to $\boxed{e}$.

(3) $a_n = \ln\left(1 + \frac{1}{n}\right)$ for $n \geq 1$

(a) $\{a_n\}$ converges ; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) = \ln(1) = 0$

(b) $\sum a_n$ diverges ;

$$a_n = \ln\left(1 + \frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$$

So we see that

$$S_n = a_1 + a_2 + \cdots + a_n = [\ln(2) - \ln(1)] + \cdots + [\ln(n+1) - \ln(n)]$$

$= \ln(n+1)$ because almost everything subtracts out

However $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = \infty$. Thus $\sum a_n$ diverges.

(4) Set $a_1 = 1$ and define a_n recursively by the rule $a_{n+1} = \frac{1}{a_n + 1}$

(a) The first five terms are

$$a_1 = 1$$

$$a_2 = \frac{1}{1+1} = 1/2$$

$$a_3 = \frac{1}{1/2 + 1} = 2/3$$

$$a_4 = \frac{1}{2/3 + 1} = 3/5$$

$$a_5 = \frac{1}{3/5 + 1} = 5/8$$

$\{a_n\}$ appears to converge to a limit

(b) Suppose $L = \lim_{n\to\infty} a_n$.

Thus $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{a_n+1} = \frac{1}{L+1}$.

Therefore $L^2 + L - 1 = 0$ and by the quadratic formula we have $L = \boxed{-1/2 + \sqrt{5}/2}$.

(c) The Fibonacci sequence $\{F_n\}$ is

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$

It is obtained by setting

$F_1 = F_2 = 1$ and defining recursevely

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 1.$$

In this problem, we have $a_n = \dfrac{F_n}{F_{n+1}}$.