HW 7 Solutions

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① Determine the values for x that make the series converge:

ⓐ $\sum \frac{x^n}{2n^2-n}$

By the ratio test, we obtain

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{2(n+1)^2-(n+1)} \cdot \frac{2n^2-n}{x^n} \right| = |x| < 1 \quad \text{So I.C. is } 1.$$

At $x=-1$, the series becomes $\sum \frac{(-1)^n}{2n^2-n}$ and converges by the A.S.T. At $x=1$, the series becomes $\sum \frac{1}{2n^2-n}$ and converges by comparing $\frac{1}{2n^2-n} \leqslant \frac{1}{n^2}$ with the convergent p-series $\sum \frac{1}{n^2}$

So we have convergence for $x \in [-1, 1]$.

ⓑ $\sum \frac{nx^n}{2^n}$

By the ratio test, we obtain

$$\lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{nx^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{2nx^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x}{2n} \right| = \frac{|x|}{2} < 1$$

So I.C. is 2. At $x=-2$, the series becomes

$\sum \frac{n(-2)^n}{2^n} = \sum n(-1)^n$ and diverges by the n-th term test. At $x=2$, the series becomes $\sum n$, and diverges also by the n-th term test. So $\sum \frac{nx^n}{2^n}$ converges for $x \in (-2, 2)$.

(g) $\sum \frac{(x-1)^n}{n^2}$

By the ratio test, we obtain

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(x-1)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)n^2}{(n+1)^2} \right| = |x-1| < 1$$

So I.C. is 1. At $x=0$, we have a convergent alternating series $\sum \frac{(-1)^n}{n^2}$. At $x=2$, we have a convergent p-series $\sum 1/n^2$. So we have convergence for $x \in [0,2]$.

(i) $\sum \frac{(x-2)^{3n}}{n!}$

By the ratio test, we obtain

$$\lim_{n\to\infty} \left| \frac{(x-2)^{3n+3}}{(n+1)!} \cdot \frac{n!}{(x-2)^{3n}} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^3}{n+1} \right| = 0 < 1$$

So we have convergence for all values of x.

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(1)

(a) $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } -1<x<1$$

Integrate:

$$\ln\left(\frac{1}{1-x}\right) + K = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad \text{for } -1<x<1$$

Plugging in $x=0$, we obtain $K=C$. Thus

$$\ln\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for } -1<x<1.$$

Now verify Eqn 6.43 in book:

$$f(x) = \ln\left(\frac{1}{1-x}\right) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{1-x} \qquad f'(0) = 1$$

$$f''(x) = \frac{1}{(1-x)^2} \qquad f''(0) = 1$$

$$f'''(x) = \frac{2}{(1-x)^3} \qquad f'''(0) = 2$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = \frac{(n-1)!}{(1-x)^n} \qquad f^{(n)}(0) = (n-1)!$$

So $\dfrac{f^{(n)}(0)}{n!} = \dfrac{1}{n}$ as desired.

(b) When $x=-1$, we have a convergent alternating series

$$\sum \frac{(-1)^n}{n}$$

so by the Remark preceeding Theorem 37

$$\log\left(\frac{1}{1-(-1)}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

In other words

$$\log 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

(3)

(a)

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (1-x)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

(c)

$$\frac{1}{3x+5} = \frac{1}{5-5(-\frac{3}{5}x)} = \frac{1}{5}\sum_{n=0}^{\infty} \left(-\frac{3}{5}x\right)^n$$

$$= \frac{1}{5}\sum_{n=0}^{\infty} \left(-\frac{3}{5}\right)^n x^n$$

(d)

$$\frac{1}{3x+5} = \frac{1}{8} \cdot \frac{1}{1-\left[-\frac{3}{8}(x-1)\right]} = \frac{1}{8}\sum_{n=0}^{\infty} \left(-\frac{3}{8}\right)^n (x-1)^n$$

(4) Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^n}$

(a) By the root test, we obtain

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{x}{n}\right)^n\right|} = \lim_{n\to\infty} \left|\frac{x}{n}\right| = 0 < 1$$

So the series converges for all x.

(b) $f(x) = x + \frac{x^2}{2^2} + \frac{x^3}{3^3} + \frac{x^4}{4^4} + \cdots\cdots$

So $\boxed{f(0) = 0}$

$$\boxed{f(1) = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots \approx 5/4}$$

* $f'(x) = 1 + 2/2^2\, x + 3/3^3\, x^2 + \cdots$

$$\boxed{f'(0) = 1}$$

$$\boxed{f'(1) = 1 + \frac{1}{2} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^4} + \cdots \approx \frac{29}{18}}$$

* $f''(x) = \frac{2}{2^2} + \frac{3 \cdot 2}{3^3}\, x + \cdots$

$$\boxed{f''(0) = 1/2}$$

(c) see part (b)