HW 9 Solutions

p478-479

(1) Find the Fourier series of the following functions:

ⓐ $f(x)=0 \quad -\pi \le x < 0; \; f(x)=1 \quad 0 \le x \le \pi$

Compute c_n:

$n=0$ | $$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-i(0)x}\,dx = \frac{1}{2\pi}\int_0^{\pi} dx = \frac{1}{2}$$

$n \neq 0$ | $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{\pi} e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\cdot\frac{1}{-in}\, e^{-inx}\Big|_0^{\pi} = \frac{-1}{2\pi i n}\left(e^{-in\pi}-1\right)$$

$$= \frac{-1}{2\pi i n}\left[(-1)^n - 1\right]$$

Write the Fourier series:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{inx} + c_{-n}e^{-inx}\right)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-1}{2\pi i n}\left[(-1)^n - 1\right]e^{inx} + \frac{-1}{2\pi i(-n)}\left[(-1)^{-n}-1\right]e^{-inx}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\left[(-1)^n - 1\right]}{2\pi i n}\left(-e^{inx} + e^{-inx}\right)$$

$$= \frac{1}{2} + \sum_{\substack{n\ge 1 \\ n \text{ odd}}} \frac{-1}{\pi i n}\cdot 2i\sin(-nx)$$

$$= \boxed{\frac{1}{2} + \sum_{\substack{n\ge 1 \\ n \text{ odd}}} \frac{2}{\pi n}\sin(nx)}$$

(b) $f(x)=0 \quad -\pi \le x < 0 \; ; \; f(x) = x \quad 0 \le x < \pi$

Compute c_n:

$\underline{n=0|}$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-i(0)x}\,dx = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{1}{2\pi}\frac{\pi^2}{2} = \frac{\pi}{4}$$

$\underline{n\neq 0|}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{\pi} x e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x}{-in} - \frac{1}{(-in)^2}\right] e^{-inx}\Big|_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{\pi}{-in} + \frac{1}{n^2}\right)e^{-in\pi} - \frac{1}{n^2}\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{\pi}{-in} + \frac{1}{n^2}\right)(-1)^n - \frac{1}{n^2}\right]$$

Write the Fourier series:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{inx} + c_{-n} e^{-inx}\right)$$

$$= \frac{\pi}{4} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left(\left[\left(\frac{\pi}{-in} + \frac{1}{n^2}\right)(-1)^n - \frac{1}{n^2}\right]e^{inx} + \left[\left(\frac{\pi}{in} + \frac{1}{n^2}\right)(-1)^{-n} - \frac{1}{n^2}\right]e^{-inx}\right)$$

$$= \frac{\pi}{4} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left[\frac{\pi(-1)^n}{in}\left(-e^{inx} + e^{-inx}\right) + \frac{(-1)^n - 1}{n^2}\left(e^{inx} + e^{-inx}\right)\right]$$

$$= \frac{\pi}{4} + \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{(-1)^n\pi}{in}\cdot 2i\sin(-nx) + \frac{(-1)^n - 1}{n^2}\,2\cos nx$$

$$= \frac{\pi}{4} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}\,2\pi}{n}\sin(nx) + \frac{2[(-1)^n - 1]}{n^2}\cos nx\right]$$

$$= \boxed{\frac{\pi}{4} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx) - \frac{2}{\pi}\sum_{\substack{n\ge 1\\ n\text{ odd}}}\frac{\cos(nx)}{n^2}}$$

(e) $F(x) = -\frac{1}{2} - \frac{x}{2\pi} \quad -\pi \le x < 0 \; ; \; F(x) = \frac{1}{2} - \frac{x}{2\pi} \quad 0 < x \le \pi \; ; \; F(0) = 0$

Compute c_n:

$n=0$ |
$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x) e^{-i(0)x}\,dx$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(-\frac{1}{2} - \frac{x}{2\pi}\right)dx + \int_{0}^{\pi}\left(\frac{1}{2} - \frac{x}{2\pi}\right)dx\right]$$
$$= \frac{1}{2\pi}\left[\frac{-\pi}{2} + \frac{\pi}{4} + \frac{\pi}{2} - \frac{\pi}{4}\right] = 0$$

$n \neq 0$ |
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x) e^{-inx}\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(-\frac{1}{2} - \frac{x}{2\pi}\right)e^{-inx}dx\right.$$
$$\left. + \int_{0}^{\pi}\left(\frac{1}{2} - \frac{x}{2\pi}\right)e^{-inx}dx\right]$$
$$= \frac{1}{2\pi}\left[\int_{0}^{-\pi}\left(\frac{1}{2} + \frac{x}{2\pi}\right)e^{-inx}dx + \int_{0}^{\pi}\left(\frac{1}{2} - \frac{x}{2\pi}\right)e^{-inx}dx\right]$$
$$= \frac{1}{2\pi}\left[-\int_{0}^{\pi}\left(\frac{1}{2} - \frac{x}{2\pi}\right)e^{inx}dx + \int_{0}^{\pi}\left(\frac{1}{2} - \frac{x}{2\pi}\right)e^{-inx}dx\right]$$
$$= \frac{1}{2\pi}\int_{0}^{\pi}\left(\frac{1}{2} - \frac{x}{2\pi}\right) 2i\sin(-nx)\,dx$$
$$= \frac{i}{2\pi}\int_{0}^{\pi}\left(\frac{x}{\pi} - 1\right)\sin(nx)\,dx$$
$$= \frac{i}{2\pi}\left[\frac{-x}{n\pi}\cos nx + \frac{1}{\pi n^2}\sin nx\right]_0^{\pi} - \frac{i}{2\pi}\int_{0}^{\pi}\sin(nx)\,dx$$
$$= \frac{i}{2\pi}\left(\frac{-1}{n}\cdot(-1)^n\right) - \frac{i}{2\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$
$$= \frac{(-1)^{n+1}i}{2\pi n} + \frac{i}{2\pi n}\left[(-1)^n - 1\right]$$
$$= \frac{i}{2\pi n}\left[(-1)^{n+1} + (-1)^n - 1\right] = \frac{-i}{2\pi n}$$

Write the Fourier Series:

$$F(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{inx} + c_{-n} e^{-inx}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{-i}{2\pi n} e^{inx} + \frac{-i}{2\pi(-n)} e^{-inx}\right)$$

$$= \sum_{n=1}^{\infty} \frac{i}{2\pi n}\left(e^{-inx} - e^{inx}\right)$$

$$= \sum_{n=1}^{\infty} \frac{i}{2\pi n} 2i \sin(-nx)$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{\sin(nx)}{\pi n}}$$

(f) $G(x) = \frac{\pi}{2} - \frac{x}{2} - \frac{x^2}{4\pi} \quad -\pi \le x \le 0 \; ; \; G(x) = \frac{\pi}{2} + \frac{x}{2} - \frac{x^2}{4\pi} \quad 0 \le x \le \pi$

Compute c_n:

$\underline{n=0}$ $$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(x) e^{-i(0)x}\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(\frac{\pi}{2} - \frac{x}{2} - \frac{x^2}{4\pi}\right)dx + \int_{0}^{\pi}\left(\frac{\pi}{2} + \frac{x}{2} - \frac{x^2}{4\pi}\right)dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{4} - \frac{\pi^2}{12} + \frac{\pi^2}{2} + \frac{\pi^2}{4} - \frac{\pi^2}{12}\right] = \frac{2\pi}{3}$$

$\underline{n \neq 0}$ $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(x) e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(\frac{\pi}{2} - \frac{x}{2} - \frac{x^2}{4\pi}\right)e^{-inx}\,dx + \int_{0}^{\pi}\left(\frac{\pi}{2} + \frac{x}{2} - \frac{x^2}{4\pi}\right)e^{-inx}\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{-\pi}\left(-\frac{\pi}{2}+\frac{x}{2}+\frac{x^2}{4\pi}\right)e^{-inx}\,dx + \int_0^{\pi}\left(\frac{\pi}{2}+\frac{x}{2}-\frac{x^2}{4\pi}\right)e^{-inx}\,dx\right]$$

$$= \frac{1}{2\pi}\left[-\int_0^{\pi}\left(-\frac{\pi}{2}-\frac{x}{2}+\frac{x^2}{4\pi}\right)e^{inx}\,dx + \int_0^{\pi}\left(\frac{\pi}{2}+\frac{x}{2}-\frac{x^2}{4\pi}\right)e^{-inx}\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi}\left(\frac{\pi}{2}+\frac{x}{2}-\frac{x^2}{4\pi}\right)e^{inx}\,dx + \int_0^{\pi}\left(\frac{\pi}{2}+\frac{x}{2}-\frac{x^2}{4\pi}\right)e^{-inx}\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left(\frac{\pi}{2}+\frac{x}{2}-\frac{x^2}{4\pi}\right)2\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left(\pi+x-\frac{x^2}{2\pi}\right)\cos nx\,dx$$

$$= \frac{1}{2\pi}\left[\left(\pi+x-\frac{x^2}{2\pi}\right)\frac{1}{n}\sin nx + \left(1-\frac{x}{\pi}\right)\frac{1}{n^2}\cos nx + \frac{1}{\pi n^3}\sin nx\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(1-\frac{\pi}{\pi}\right)\frac{1}{n^2}\cos(n\pi) - \left(1-\frac{0}{\pi}\right)\frac{1}{n^2}\cos(0)\right]$$

$$= \frac{-1}{2\pi n^2}$$

Write the Fourier series:

$$G(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{inx} + c_{-n}e^{-inx}\right)$$

$$= \frac{2\pi}{3} + \sum_{n=1}^{\infty}\frac{-1}{2\pi n^2}e^{inx} + \frac{-1}{2\pi(-n)^2}e^{-inx}$$

$$= \frac{2\pi}{3} + \sum_{n=1}^{\infty}\frac{-1}{2\pi n^2}\cdot 2\cos nx$$

$$= \boxed{\frac{2\pi}{3} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}}$$

⑤ Let $f(x)=0 \quad -\pi \le x < 0$; $f(x)=1 \quad 0 \le x \le \pi$

By Problem 1a, $f(x) \sim \frac{1}{2} + \sum_{\substack{n \ge 1 \\ n \text{ odd}}} \frac{2}{\pi n} \sin(nx)$

So for $0 < x < \pi$, we have

$$1 = \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right)$$

When $x = \pi/2$, we have

$$1 = \frac{1}{2} + \frac{2}{\pi}\left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \cdots \right)$$

$$1 = \frac{1}{2} + \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right)$$

$$\Rightarrow \frac{1}{2} = \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

⑥

ⓐ Plug in $x = \frac{\pi}{4}$ in Problem 1a, you get

$$1 = \frac{1}{2} + \frac{2}{\pi}\left(\sin\frac{\pi}{4} + \frac{1}{3}\sin\frac{3\pi}{4} + \frac{1}{5}\sin\frac{5\pi}{4} + \cdots \right)$$

$$1 = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sqrt{2}}{2}\right)\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots\right)$$

$$1 = \frac{1}{2} + \frac{\sqrt{2}}{\pi}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots\right)$$

$$\Rightarrow \frac{1}{2} = \frac{\sqrt{2}}{\pi}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots\right)$$

$$\Rightarrow \frac{\pi}{\sqrt{8}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots$$

(b) Plug in $x=0$ in Problem 1b (Plugging in $x=\pm\pi$ also works)

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{1}{n^2} \Rightarrow \frac{\pi}{4} = \frac{2}{\pi}\sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{1}{n^2}$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{\substack{n\geq 1 \\ n \text{ odd}}} \frac{1}{n^2}$$

(c) Plug in $x=0$ in Problem 1d.

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

(d) Plug in $x=0$ in Problem 1f (or $x=\pi$ in Problem 1d)

Plugging in $x=0$ in Problem 1f yields

$$G(0) = \frac{\pi}{2} = \frac{2\pi}{3} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow -\frac{\pi}{6} = \frac{-1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Plugging in $x=\pi$ in Problem 1d yields

$$f(\pi) = \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$