Math 8 - Midterm 2 Solutions

Fall, 2007

1. Give an example of a function with the stated property, or briefly explain why no such function can exist.

   (a) A surjective function $f : A \to B$ that is not injective. (Please also specify the sets A and B.)

   Solution. Let $A = \{1, 2\}$ and $B = \{3\}$, and define $f : A \to B$ by $f(1) = f(2) = 3$. This f is surjective since all elements of B, namely just 3, are images of elements of A. But f is not injective since $f(1) = f(2)$.

   (b) A one-to-one function $f : \{0, 1, 2\} \to \{3, 4\}$.

   Solution. No such function can exist. For f to be one-to-one each of the three elements $0, 1, 2$ must be sent to a different element of the set $\{3, 4\}$. However, this set has only two elements so this is impossible.

2. Let $P(a, b)$ stand for the proposition "a knows b's name," and assume the universe of discourse is the set of all people. Write the following statements symbolically.

   (a) "Each person knows the names of at least two people."

   Solution. $\forall a\ \exists b\ \exists c\ (P(a, b) \wedge P(a, c))$

   (b) "Somebody knows only their own name and no others."

   Solution. $\exists a\ \forall b\ [P(a, a) \wedge (P(a, b) \Rightarrow b = a)]$

3. True or False? Give brief justifications for your answers.

   (a) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{Z}\ (x < y)$.

   Solution. True. This just says that for any real number x there is an integer y that is larger than x. This is obvious.

   (b) $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z}\ \sim(2|xy)$. (Recall that $a|b$ means b is an integer multiple of a.)

   Solution. False. Notice first that $\sim(2|xy)$ means that xy is not a multiple of 2, or in other words, that xy is odd. Thus the proposition says that there is an integer x such that xy is odd for all integers y. But this is impossible, since no matter what x is, $2x$ will be even.

4. Let A and B be sets. Prove that $A \subseteq B$ if and only if $A \cap B = A$.

   Solution. We first prove $A \subseteq B \Rightarrow A \cap B = A$. Assume $A \subseteq B$. We must show that $A \cap B \subseteq A$ and $A \subseteq A \cap B$. It is always true that $A \cap B \subseteq A$ (If x is an element of both A and B, then it is an element of A). Now let $x \in A$. Since we are assuming $A \subseteq B$, we know that $x \in B$. Thus, for any x, we have

$$x \in A \Rightarrow (x \in A \wedge x \in B) \equiv x \in A \cap B.$$

Hence we have shown that $A \subseteq A \cap B$. Together, these two inclusions prove that $A = A \cap B$.

We now prove that $A \cap B = A \Rightarrow A \subseteq B$. Assume that $A \cap B = A$. Then $A = A \cap B \subseteq B$ shows that $A \subseteq B$.

5. For each $n \in \mathbb{Z}$, let
$$A_n = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a + b = n\}.$$

   (a) Write down A_0, A_1 and A_{-3} by listing (some of) their elements between braces.
   Solution. $A_0 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a + b = 0\} = \{\ldots, (-1, 1), (0, 0), (1, -1), (2, -2), \ldots\}$.
   $A_1 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a + b = 1\} = \{\ldots, (-1, 2), (0, 1), (1, 0), (2, -1), \ldots\}$.
   $A_{-3} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a + b = -3\} = \{\ldots, (-1, -2), (0, -3), (1, -2), (2, -1), \ldots\}$.

   (b) What is $\bigcup_{n \in \mathbb{Z}} A_n$?
   Solution. $\bigcup_{n \in \mathbb{Z}} A_n$ is the set of all ordered pairs (x, y) that appear in at least one of the sets A_n for some integer n. But any ordered pair (x, y) of integers appears in the set A_{x+y}. Thus the union of all the sets A_n will be all of $\mathbb{Z} \times \mathbb{Z}$.