Math 8 - Final Exam Review Problems
Winter 2007

While the final exam will focus on topics covered since the second midterm (Ch. 2.10, 3.2-3.3, 4.1-4.3), you will still be expected to know all the material from chapters 1 and 2 as well. The MOST IMPORTANT topics from these chapters are:

- Conditional and Biconditional statements (Ch. 1.3-1.4). You should be able to translate between words and symbols. Know how to formulate Converses and Contrapositives.
- Proof Techniques (Ch. 1.4): You should know how to formulate Direct Proofs, Indirect Proofs, and Proofs by Contradiction.
- Quantifier Notation (Ch. 2.3). You should be able to translate between words and symbols, and also recognize the truth value of propositions expressed with quantifiers.
- Sets and Notation (eg. $\in, \subseteq$, Ch. 2.1, 2.4).
- Set Equality and Inclusion (Ch. 2.4). Be able to prove that one set is a subset of another, or that two sets are equal. The sets in question may be given by set operations (as in the next two items), so you must know how to use the definitions of these.
- Unions, Intersections, and Complements. (Ch. 2.5). Know the definitions! Be able to illustrate these sets using Venn diagrams.
- Power Sets (Ch. 2.7) and Cartesian Products (Ch. 2.8). Know the definitions!

New Topics

1. **Functions.** (a)-(c) Give examples of the following, or briefly explain why no example exists.

   (a) An injection $f : \mathbb{N} \to \mathbb{N}$ that is not surjective.

   (b) An injection $f : \mathbb{N} \to [0, 1]$.

   (c) An injection $f : A \to B$ and a surjection $g : B \to C$ such that $g \circ f$ is not injective.

   (d) True or False: Let A and B be sets, and suppose $f : A \to B$ is an injection. Then there exists a surjection $g : B \to A$. Give a proof or counterexample.

   (e)-(g) Determine whether the following functions are one-to-one, onto, or both. Justify your answers. (Good Practice: For each function, write out the statements "f is one-to-one", "f is onto", etc. symbolically using the definitions.)

   (e) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined by $f((a, b)) = a + b$ for all $a, b \in \mathbb{Z}$.

   (f) $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is defined by $g(n) = (n, -n)$ for all $n \in \mathbb{Z}$.

(g) $h : \mathcal{P}(\mathbb{N}) - \{\emptyset\} \to \mathbb{N}$ is defined by $g(S) = \min S$, the smallest element of S, for any nonempty $S \subseteq \mathbb{N}$.

(h) Let S be a nonempty set. Show that the function $F : \mathcal{P}(S) \to \mathcal{P}(S)$, defined by $F(A) = S - A$ for any $A \subseteq S$, is bijective, and describe the inverse function F^{-1}. (Hint: one way to show that F is bijective is to first find the inverse function and show that the compositions in both orders $F \circ F^{-1}$ and $F^{-1} \circ F$ are the identity functions.)

2. **Cardinality (4.1-4.3).** You are only responsible for the material in the following pages: **4.1**: 151-156, 158-159, **4.2**: 169-171, **4.3**: 176-177, 181-184. Most of this was also covered in lecture (although without the proofs).

   You MUST KNOW: Definitions 4.1, 4.7, 4.27, and 4.33. Theorems (statements only): 4.8, 4.9, 4.26, 4.39, 4.40, 4.41.

   You SHOULD ALSO KNOW: Theorems (statements only): 4.11-13 (obvious, but good to know), 4.35-37. These may be helpful in writing proofs or coming up with examples. Understanding the proofs of all these theorems is not essential, but can be helpful in writing your own proofs or thinking of examples.

   On the test you will be allowed to use any of these theorems as facts, without providing justification. For example, to justify that there is no injection from $\{1, 2\}$ to $\{4\}$, you could simply state "If there is an injection from A to B then $|A| \leq |B|$, but here $|A| = 2$ and $|B| = 1$." (The fact used here is just the contrapositive of 4.9.)

   (a)-(c) Give examples or explain why no examples exist.

   (a) A surjection $f : \mathbb{N}_n \to \mathbb{N}_n$ that is not injective. (Recall $\mathbb{N}_n = \{1, 2, 3, \ldots, n\}$.)

   (b) An injection $f : \mathbb{R} \to \mathbb{N}$.

   (c) A surjection $f : \mathbb{R} \to \mathbb{N}$. It may be easier to just describe (in words or a graph) a rule defining this function, without giving a formula.

   (d) Suppose $A \approx C$ and $B \approx D$. Prove that $A \times B \approx C \times D$.

3. **Induction.**

   (a) Prove that for any real number $x \geq -1$ and any integer $n \geq 1$,

$$(1 + x)^n \geq 1 + nx.$$

   (b) Prove that for any integer $n \geq 1$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$