Math 8 - Solutions to Final Exam Review Problems

Fall 2007

1. **Functions.** (a)-(c) Give examples of the following, or briefly explain why no example exists.

   (a) An injection $f : \mathbb{N} \to \mathbb{N}$ that is not surjective.
   Solution. Let $f(n) = n + 1$. This is injective since for any $a, b \in \mathbb{N}$, $f(a) = f(b)$ implies $a + 1 = b + 1$, which implies $a = b$. This is not surjective since for all $a \in \mathbb{N}$, $f(a) \neq 1$.

   (b) An injection $f : \mathbb{N} \to [0, 1]$.
   Solution. Let $f(n) = 1/n$. This is injective since for any $a, b \in \mathbb{N}$, $f(a) = f(b)$ implies $1/a = 1/b$, which implies $a = b$.

   (c) An injection $f : A \to B$ and a surjection $g : B \to C$ such that $g \circ f$ is not injective.
   Solution. Let $A = \{0, 1\}$, $B = \{2, 3, 4\}$ and $C = \{5\}$. Define f by the set of ordered pairs $\{(0, 2), (1, 3)\}$ (ie., $f(0) = 2$ and $f(1) = 3$), and define g by the set of ordered pairs $\{(2, 5), (3, 5), (4, 5)\}$ (this is the only function from B to C here). Clearly f is injective since $f(0) \neq f(1)$, g is surjective since $g(2) = 5$, but $g \circ f$ is not injective since $g(f(0)) = 5 = g(f(1))$.

   (d) True or False: Let A and B be sets, and suppose $f : A \to B$ is an injection. Then there exists a surjection $g : B \to A$. Give a proof or counterexample.
   Solution. FALSE! This is not true if $A = \emptyset$. Any function $f : \emptyset \to B$ (in fact there is only one) is automatically one-to-one, since in order not to be one-to-one there must be two elements of $\emptyset$ that produce the same output. However, there are NO functions $g : B \to \emptyset$ whenever $B \neq \emptyset$, since there are no possible outputs in $\emptyset$ for the elements of B.
   However, if A is assumed to be nonempty, it is TRUE. A surjection g can be constructed as follows. If $b = f(a)$ for some $a \in A$ then this a is unique (since f is one-to-one) and we may define $g(b) = a$. Otherwise, define $g(b) = a_0$ where a_0 is some fixed element of A. If $a \in A$, then $a = g(f(a))$ by definition of g, so g is surjective.

   (e)-(g) Determine whether the following functions are one-to-one, onto, or both. Justify your answers. (Good Practice: For each function, write out the statements "f is one-to-one", "f is onto", etc. symbolically using the definitions.)

   (e) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined by $f((a, b)) = a + b$ for all $a, b \in \mathbb{Z}$.
   Solution. f is onto: if $n \in \mathbb{Z}$, then $n = n+0 = f((n, 0))$. But f is not one-to-one: $f(0, 0) = 0 = f(1, -1)$.

   (f) $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is defined by $g(n) = (n, -n)$ for all $n \in \mathbb{Z}$.
   Solution. g is one-to-one: if $g(n) = g(m)$, then $(n, -n) = (m, -m)$, and thus $n = m$. But g is not onto: If $(0, 1) = g(n)$ for some $n \in \mathbb{Z}$, then $(0, 1) = (n, -n)$ so we must have $n = 0$ and $-n = 1$, which implies $0 = n = -1$, a contradiction.

(g) $h : \mathcal{P}(\mathbb{N}) - \{\emptyset\} \to \mathbb{N}$ is defined by $g(S) = \min S$, the smallest element of S, for any nonempty $S \subseteq \mathbb{N}$.

Solution. h is onto: If $n \in \mathbb{N}$, n is the smallest element in the set $\{n\}$, so $n = h(\{n\})$. But h is not one-to-one since $h(\{1, 2\}) = 1 = h(\{1\})$.

(h) Let S be a nonempty set. Show that the function $F : \mathcal{P}(S) \to \mathcal{P}(S)$, defined by $F(A) = S - A$ for any $A \subseteq S$, is bijective, and describe the inverse function F^{-1}. (Hint: one way to show that F is bijective is to first find the inverse function and show that the compositions in both orders $F \circ F^{-1}$ and $F^{-1} \circ F$ are the identity functions.)

Solution. Notice that $F(A)$ is just the complement of the subset A in S. In order to get the original subset A back from its complement, we just need to take the complement again. This suggests that $F^{-1}(A) = F(A) = S - A$ for any $A \subseteq S$. Indeed, we have $F^{-1}(F(A)) = S - (S - A) = A = F(F^{-1}(A))$ for any $A \subseteq S$. Thus, we know that F is a bijection.

2. **Cardinality (4.1-4.3).** (a)-(c) Give examples or explain why no examples exist.

(a) A surjection $f : \mathbb{N}_n \to \mathbb{N}_n$ that is not injective. (Recall $\mathbb{N}_n = \{1, 2, 3, \dots, n\}$.)

Solution. No examples exist. If f is not injective, there are two different integers that are mapped to the same image. That leaves $n-2$ remaining inputs in $\mathbb{N}$ and $n-1$ remaining outputs in $\mathbb{N}$. Since $n-2 < n-1$, not every possible output can be the image of one of these inputs, so f cannot be surjective.

(b) An injection $f : \mathbb{R} \to \mathbb{N}$.

Solution. No such injection can exist. We know that $\mathbb{R}$ is uncountable, while $\mathbb{N}$ is countable, and every subset of a countable set is countable. However, if $f : \mathbb{R} \to \mathbb{N}$ is injective, it induces a bijection between $\mathbb{R}$ and its image, which is a subset of $\mathbb{N}$. Thus we would have a subset of $\mathbb{N}$ that is uncountable, but this is impossible.

(c) A surjection $f : \mathbb{R} \to \mathbb{N}$. It may be easier to just describe (in words or a graph) a rule defining this function, without giving a formula.

Solution. f can be defined using the *greatest integer function* $[x]$ (see p. 123), which rounds a real number down to the nearest integer. Since the image of f must be a natural number, we should first take the absolute value of $x \in \mathbb{R}$, then round down to the nearest integer, and finally add 1 (so we don't end up with 0). In symbols, $f(x) = [|x|] + 1$.

(d) Suppose $A \approx C$ and $B \approx D$. Prove that $A \times B \approx C \times D$.

Solution. Assume $A \approx C$ and $B \approx D$. This means that we have bijections $f : A \to C$ and $g : B \to D$. Define $h : A \times B \to C \times D$ by $h(a, b) = (f(a), g(b))$ for all $a \in A$ and $b \in B$. We check that h is one-to-one and onto.

One-to-one: Suppose $h(a, b) = h(a', b')$. This means that $(f(a), g(b)) = (f(a'), g(b'))$, which implies that $f(a) = f(a')$ and $g(b) = g(b')$. Since f and g are one-to-one, we can conclude that $a = a'$ and $b = b'$. Hence $(a, b) = (a', b')$.

Onto: Let $(c, d) \in C \times D$. Since f and g are onto, there exist $a \in A$ and $b \in B$ such that $f(a) = c$ and $g(b) = d$. Thus $h(a, b) = (f(a), g(b)) = (c, d)$.

3. **Induction.**

(a) Prove that for any real number $x \geq -1$ and any integer $n \geq 1$,

$$(1 + x)^n \geq 1 + nx.$$

Solution. We prove the proposition by induction on $n \geq 1$. Fix a real number $x \geq -1$, and let $P(n)$ be the proposition $(1 + x)^n \geq 1 + nx$.
Basis Step. Let $n = 1$. $P(1)$ says $1 + x \geq 1 + x$, which is clearly true.
Inductive Step. Let $k \geq 1$, and assume $P(K) : (1+x)^k \geq 1+kx$. We must prove $P(k + 1) : (1 + x)^{k+1} \geq 1 + (k + 1)x$.

$$\begin{aligned}
(1 + x)^{k+1} &= (1 + x)^k(1 + x) \\
&\geq (1 + kx)(1 + x) \quad \text{(by P(k) and since } 1 + x \geq 0) \\
&= 1 + kx + x + x^2 \\
&\geq 1 + (k + 1)x \quad (\text{since } x^2 \geq 0).
\end{aligned}$$

Thus, $(1 + x)^{k+1} \geq 1 + (k + 1)x$, and by induction $(1 + x)^n \geq 1 + nx$ holds for all $n \geq 1$.

(b) Prove that for any integer $n \geq 1$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Solution. We prove the proposition by induction on $n \geq 1$. Basis Step. Let $n = 1$. We must show $\frac{1}{1^2} \leq 2 - \frac{1}{1}$, but this is clear since both sides of the inequality are equal to 1.
Inductive Step. Let $k \geq 1$, and assume

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}.$$

We now prove that the same inequality holds for $k + 1$:

$$\begin{aligned}
\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(k+1)^2} &= \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2}\right) + \frac{1}{(k+1)^2} \\
&\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\
&\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)} \\
&= 2 - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1} \\
&= 2 - \frac{1}{k+1}.
\end{aligned}$$

Thus the given inequality holds for all $n \geq 1$ by induction.