Solutions to Midterm 2 Review Problems

Math 8, Fall 2007

1. Write the following sets in the form a) $\{x \in S \mid P(x)\ \}$ and the form b) $\{f(x) \mid x \in S\ \}$.

   (i) $\{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \ldots\}$

   (ii) $\{11, 21, 31, 41, \ldots\}$

   Solution. (i) a) $\{x \in \mathbb{R} \mid x^2 \in \mathbb{N} \wedge x > 0\}$; b) $\{\sqrt{x} \mid x \in \mathbb{N}\}$

   (ii) a) $\{x \in \mathbb{N} \mid 10|(x-1)\}$; b) $\{10x + 1 \mid x \in \mathbb{N}\}$

2. List (or otherwise describe) the elements of the set $\{5x - 1 \mid x \in \mathbb{Z}\}$.

   Solution. The elements of this set are all the integers that are one less than a multiple of 5; i.e., it equals the set $\{\ldots, -6, -1, 4, 9, \ldots\}$.

3. Prove that $\{5x - 1 \mid x \in \mathbb{Q}\} = \mathbb{Q}$.

   Solution. We will show that $\{5x - 1 \mid x \in \mathbb{Q}\} \subseteq \mathbb{Q}$ and that $\mathbb{Q} \subseteq \{5x - 1 \mid x \in \mathbb{Q}\}$. (This is all we need to show since we know that for any two sets A and B: $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$.)

   $\{5x - 1 \mid x \in \mathbb{Q}\} \subseteq \mathbb{Q}$: Let $y \in \{5x - 1 \mid x \in \mathbb{Q}\}$. This means that $y = 5x - 1$ for some rational number x. Clearly such a y must also be rational (if $x = a/b$ for $a, b \in \mathbb{Z}$, then $y = 5x - 1 = (5a - b)/b$). Hence $y \in \mathbb{Q}$.

   $\mathbb{Q} \subseteq \{5x - 1 \mid x \in \mathbb{Q}\}$: Let $y \in \mathbb{Q}$. We want to show that $y = 5x - 1$ for some $x \in \mathbb{Q}$. If this were true, we can see that $x = (y + 1)/5$. So let $x = (y + 1)/5$ and note that $x \in \mathbb{Q}$ (if $y = a/b$ for $a, b \in \mathbb{Z}$, then $x = (y + 1)/5 = (a + b)/5b$). Hence $y = 5x - 1$ with $x = (y + 1)/5 \in \mathbb{Q}$, showing that y is an element of the second set too.

4. True or False? Give brief justifications. (The universe of discourse is $\mathbb{R}$.)

   (a) $\forall x\ \forall y\ (xy \geq 0)$

   (b) $\exists a\ \forall b\ (ba = b/a)$

   Solution. (a) This says that the product of any two real numbers x and y is nonnegative. This is FALSE! For instance, $-1 * 1 = -1 < 0$. In fact, any two numbers x and y whose product is negative (eg. one is positive and the other is negative) provide a counterexample.

   (b) This says that there is a real number a such that $ba = b/a$ is always true. This is TRUE! Simply let $a = 1$, as $b * 1 = b/1$ for all $b \in \mathbb{R}$. $a = -1$ would also work.

5. Let $P(x, y)$ stand for the proposition "x and y are friends", and assume the universe of discourse is the set of all people. Express the following propositions symbolically:

   (a) "Every two people have a common friend."

   (b) "Nobody is friends with everyone."

   Solution. (a) $\forall x\ \forall y\ \exists z\ (P(x, z) \wedge P(y, z))$

(b) $\sim \exists x\ \forall y\ P(x,y)$. This is also equivalent to $\forall x\ \exists y\ \sim P(x,y)$, which corresponds to the equivalent (but somewhat ambiguous) statement "Everyone is not friends with someone."

6. Shade in the region corresponding to $S = A \cap (B \cup C)$ on a Venn diagram. On a separate diagram, shade in the region corresponding to $T = A \cup (B \cap C)$. If these are different, give an example of sets A, B, C and an element x that belongs to one of the sets S, T but not the other.

   Solution. No pictures here, but the shaded regions are definitely different. The first only contains that part of A that is overlapped by either B or C, while the second contains all of A. For the example, let $A = \{1,2\}, B = \{2,3\}$ and $C = \{2,4\}$. Then $S = \{2\}$, while $T = \{1,2\}$, so $S \neq T$.

7. Let A and B be sets. Prove that $A \cap B = A \cup B$ if and only if $A = B$.

   Solution. $\Rightarrow$: Assume that $A \cap B = A \cup B$. We will show that $A \subseteq B$ and $B \subseteq A$, since this is equivalent to $A = B$. We have the following subset inclusions

$$A \subseteq A \cup B = A \cap B \subseteq B$$

   and

$$B \subseteq A \cup B = A \cap B \subseteq A,$$

   from which we obtain $A \subseteq B$ and $B \subseteq A$.

   $\Leftarrow$: Assume $A = B$. Then $A \cap B = A \cap A = A$ and $A \cup B = A \cup A = A$. So $A \cap B = A = A \cup B$.

8. For each $i \in \mathbb{N}$, let $A_i = \{x \in \mathbb{N} \mid x \geq i\ \}$. Compute $\bigcup_{i\in\mathbb{N}} A_i$ and $\bigcap_{i\in\mathbb{N}} A_i$. (Give brief justifications for your answers, but not rigorous proofs.)

   Solution. We begin by writing down the sets A_i for a few values of i.

   $A_1 = \{x \in \mathbb{N} \mid x \geq 1\} = \{1, 2, 3, \ldots\}$

   $A_2 = \{x \in \mathbb{N} \mid x \geq 2\} = \{2, 3, 4, \ldots\}$

   $A_3 = \{x \in \mathbb{N} \mid x \geq 3\} = \{3, 4, 5, \ldots\}$

   $\bigcup_{i\in\mathbb{N}} A_i$ is the set of all numbers that belong to at least one A_i. Since all natural numbers already belong to A_1, this union will be $\mathbb{N}$.

   $\bigcap_{i\in\mathbb{N}} A_i$ is the set of all numbers that belong to all of the A_i's. But as i grows, each set A_i excludes more numbers (namely, A_i excludes all the numbers less than i), and we can see that no natural number will be contained in *every* A_i. For instance, $n \notin A_{n+1} = \{n+1, n+2, n+3, \ldots\}$. Thus, the intersection is the empty set $\emptyset$.

9. What is $\mathcal{P}(\{a, \{b, c\}\})$?

   Solution. $\mathcal{P}(\{a, \{b, c\}\}) = \{\emptyset, \{a\}, \{\{b, c\}\}, \{a, \{b, c\}\}\}$

10. Is it always true that $A \cap \mathcal{P}(A) = \emptyset$? Provide a justification or a counterexample.

Solution. No! The simplest way to find an example of a set A such that $A \cap \mathcal{P}(A) \neq \emptyset$ is to notice that $\mathcal{P}(A)$ always contains $\emptyset$ as an element. Thus if A is any set containing the empty set $\emptyset$ as an element, $\emptyset$ will also be an element of the intersection $A \cap \mathcal{P}(A)$. Since this intersection has an element, it is nonempty.

For example, if $A = \{\emptyset\}$, then $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}\}$, and $A \cap \mathcal{P}(A) = \{\emptyset\}$. Another example can be obtained by letting $A = \{a, \{a\}\}$, since then $\{a\}$ is also an element of $\mathcal{P}(A)$.

11. (a) Give an example of a subset of $\{a, b, c\} \times \{a, b, c\}$ that is not equal to $A \times B$ for any subsets A, B of $\{a, b, c\}$.

(b) Give an example of a subset of the plane $\mathbb{R}^2$ (a picture will suffice) that is not of the form $A \times B$ for subsets A, B of $\mathbb{R}$. Justify briefly.

Solution. (a) $S = \{(a, b), (b, a)\}$ works. If this were the cartesian product $A \times B$ for two subsets A and B of $\{a, b, c\}$, then all the first coordinates would have to belong to A and all the second coordinates would have to belong to B. This means that $A = \{a, b\}$ since both a and b are first coordinates of elements of S, and similarly $B = \{a, b\}$. But then $A \times B$ would also contain (a, a), which is not in S.

(b) Any set that consists of all the points on some slanted line will work, or the set of all the points on a circle, and there are many more possibilities. The important thing is that not every possible first coordinate occurs in an ordered pair with every possible second coordinate. Also, the set S from part (a) works, as long as a and b are any two distinct real numbers.

12. Give examples of sets A, B, C and functions $f : A \to B$ and $g : B \to C$ such that (Note: each letter represents a separate problem.)

(a) f is surjective, but not bijective.

(b) g is surjective, but $g \circ f$ is not surjective.

(c) $g \circ f$ is bijective, but f is not bijective.

You may give your answers in the form of tables, graphs or function diagrams, but be sure to label everything appropriately.

Solution. We will write our answers as sets of ordered pairs.

(a) Let $A = \{1, 2\}, B = \{3\}$ and $f = \{(1, 3), (2, 3)\}$. Then f is surjective because $f(1) = 3$ and 3 is the only element of B. f is not bijective since $f(1) = 3 = f(2)$ (so f is not injective).

For the next two examples we will use the sets $A = \{1, 2\}, B = \{3, 4, 5\}, C = \{6, 7\}$.

(b) Let $g = \{(3, 6), (4, 7), (5, 7)\}$ and $f = \{(1, 4), (2, 5)\}$. Then g is surjective since $g(3) = 6$ and $g(4) = 7$, but $g \circ f = \{(1, 7), (2, 7)\}$ is not surjective since no element of A is sent to 6.

(c) Let $f = \{(1,4),(2,5)\}$ and $g = \{(3,6),(4,6),(5,7)\}$. Then f is not bijective since it is not surjective: no element of A is sent by f to 3. But $g \circ f = \{(1,6),(2,7)\}$ is bijective since it sends exactly one element of A to each element of C.