Math 8 - Midterm Review Problems
Winter 2007

1. Prove the logical equivalence:

$$(P \wedge \sim Q) \wedge (R \Rightarrow Q) \equiv \sim [(P \Rightarrow Q) \vee R].$$

2. Simplify the sentential form $(P \wedge \sim Q) \Rightarrow (P \vee Q)$ as much as possible.

3. Write the following propositions symbolically with no words. (You do not have to prove them.)

   (a) "There does not exist a largest real number."

   (b) "The interval strictly between any two distinct real numbers contains at least one rational number."

   (c) "Every nonempty set has at least two distinct subsets."

4. Determine whether the following statements are true or false, where the universe of discourse is the set of all real numbers, and give a brief justification.

   (a) $\forall x\ \exists y\ [(y > 0) \Rightarrow (xy > 0)]$

   (b) $\forall x\ \exists y\ \forall z\ [(x + y)z^2 \leq 0]$

   (c) $\exists x\ \forall y\ (xy = 1)$

   (d) $\forall y\ \exists x\ (x < y < x + 1)$

5. Recall that the *Sheffer stroke* of two propostions P and Q is defined as

$$P \uparrow Q \equiv \sim (P \wedge Q).$$

   If $A = \{x \mid P(x)\}$ and $B = \{x \mid Q(x)\}$, let $S = \{x \mid P(x) \uparrow Q(x)\}$. (Assume everything is contained in a fixed domain of interpretation U.)

   (a) Describe the set S in terms of A and B, using the standard set operations (eg. union, intersection, set difference, etc.).

   (b) Illustrate S using a Venn Diagram.

   (c) If we also know that $A \subseteq B$, what else can we say about S?

6. Let A be a finite set, and let B be a subset of A. Prove that $A = B$ if and only if $|A| = |B|$. (Recall, $|A|$ is the cardinality of A, i.e., the number of elements of A.)

7. Let A, B, C be sets. Prove:

   (a) If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

   (b) If $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

8. Consider the proposition: "Every nonzero rational number is equal to a product of two irrational numbers."

   (a) Write this proposition using only symbols and no words.

   (b) Prove this proposition.

9. Consider the family $\{A_n\}_{n\in\mathbb{N}}$ of subsets

$$A_n = \{x \in \mathbb{R} \mid nx \in \mathbb{Z}\}$$

   of $\mathbb{R}$, indexed by the set $\mathbb{N}$ of natural numbers. Prove:

   (a) $\displaystyle\bigcup_{n\in\mathbb{N}} A_n = \mathbb{Q}$.

   (b) $\displaystyle\bigcap_{n\in\mathbb{N}} A_n = \mathbb{Z}$.