STABLE EQUIVALENCES OF GRADED ALGEBRAS

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ABSTRACT. We extend the notion of stable equivalence to the class of locally finite graded algebras. For such an algebra Λ, we focus on the Krull-Schmidt category gr_Λ of finitely generated $\mathbb{Z}$-graded Λ-modules with degree 0 maps, and the stable category $\underline{\mathrm{gr}}_\Lambda$ obtained by factoring out those maps that factor through a graded projective module. We say that Λ and Γ are *graded stably equivalent* if there is an equivalence $\alpha : \underline{\mathrm{gr}}_\Lambda \stackrel{\approx}{\to} \underline{\mathrm{gr}}_\Gamma$ that commutes with the grading shift. Adapting arguments of Auslander and Reiten involving functor categories, we show that a graded stable equivalence α commutes with the syzygy operator (where defined) and preserves finitely presented modules. As a result, we see that if Λ is right noetherian (resp. right graded coherent), then so is any graded stably equivalent algebra. Furthermore, if Λ is right noetherian or k is artinian, we use almost split sequences to show that a graded stable equivalence preserves finite length modules. Of particular interest in the nonartinian case, we prove that any graded stable equivalence involving an algebra Λ with soc $\Lambda = 0$ must be a graded Morita equivalence.

Understanding where and how stable equivalences arise between algebras poses an important, albeit difficult, problem in the represention theory of artin algebras and finite groups. To address this problem it is natural to look for clues by considering stable module categories that arise in other contexts where they admit alternative descriptions. In the most notable examples, the stable categories that appear are quotients of categories of *graded* modules. Such categories appear, for instance, in the work of Bernstein, Gel'fand, and Gel'fand on the bounded derived category of coherent sheaves on projective n-space [8], and in the work of Orlov generalizing these results to certain noncommutative projective varieties [24]. Stable categories of graded modules are also important in the theory of Koszul duality for self-injective Koszul algebras [22], and in the study of derived equivalences for finite-dimensional algebras of finite global dimension [13]. It thus appears worthwhile to generalize the classical theory of stable equivalences of finite-dimensional algebras to categories of graded modules over graded algebras.

In this paper we extend the functor category methods of Auslander and Reiten to study stable equivalences between categories of graded modules over a wide class of graded algebras which are not necessarily noetherian. Throughout, we fix a commutative semilocal noetherian ring k that is complete with respect to its Jacobson radical $\mathfrak{m}$, and focus on *nonnegatively graded, locally finite k-algebras*, by which we mean those graded k-algebras $\Lambda = \oplus_{i\geq 0}\Lambda_i$ where each Λ_i is a finitely generated k-module. Notice that if $\Lambda = \Lambda_0$ we recover precisely the class of *noetherian algebras* arising, for example, in the integral representation theory of finite groups. Alternatively, if k is artinian, we obtain the class of locally artinian graded algebras studied in [20], which includes graded quotients of path algebras of quivers. In particular, our theory applies to the preprojective algebras as well as quantum polynomial rings arising in noncommutative alegebraic geometry.

We shall work primarily with the category gr_Λ of finitely generated $\mathbb{Z}$-graded right Λ-modules and degree 0 morphisms. As usual, the graded stable category of Λ is defined as the quotient category $\underline{\mathrm{gr}}_\Lambda$ obtained from gr_Λ by factoring out all maps that factor through a graded projective module. This category inherits a self-equivalence given by the grading shift, and we shall say that two algebras Λ and Γ are *graded stably equivalent* if there is an equivalence $\alpha : \underline{\mathrm{gr}}_\Lambda \stackrel{\approx}{\longrightarrow} \underline{\mathrm{gr}}_\Gamma$ that commutes with the grading shift. While the module categories we consider are typically not dualizing R-varieties in the sense of [5] (since the duality does not preserve finitely presented modules), they are still Krull-Schmidt categories. As a consequence, our proofs

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often avoid use of the duality and easily generalize to stable equivalences between abelian Krull-Schmidt categories with enough projectives.

The first examples of nontrivial graded stable equivalence for nonartinian algebras can be found in [21], where it is observed that the processes of constructing and separating nodes work just as nicely for infinite-dimensional graded factors of path algebras of quivers. In developing the theory of invariants of graded stable equivalences below, one of our main motivations is to determine the nature of other examples. In this direction, the main result of this paper is essentially a non-existence result, stating that there are no nontrivial graded stable equivalences involving an algebra Λ with zero socle. The proof of this is given in Section 8, after much preparation. Along the way we find many interesting features of graded algebras that are determined by the stable category. In Section 3, for instance, we show that the property of being right noetherian can be detected in $\underline{\mathrm{gr}}_\Lambda$. The proof requires a close study of the effect of a stable equivalence on a class of extensions. We will also see that the subcategories of finitely presented modules correspond, and that on finitely presented modules, a stable equivalence commutes with the syzygy operator. Using this result, we extend our analysis of short exact sequences and show that a graded stable equivalence induces an isomorphism of stable Grothendieck groups. We also interpret these results in terms of Ext-groups, establishing isomorphisms between them in certain cases. We conclude with another example of graded stable equivalence, obtained by modifying a construction of Liu and Xi. However, it is still largely open as to what types of more interesting examples may arise.

1. Notation and preliminaries

All algebras we will consider in this article are assumed to be positively graded, locally finite k-algebras for a commutative semilocal noetherian ring k, complete with respect to its Jacobson radical $\mathfrak{m}$, unless noted otherwise. Recall that a nonnegatively graded k-algebra $\Lambda = \oplus_{i \geq 0} \Lambda_i$ is *locally finite* if each Λ_i is finitely generated over k and $k \subseteq Z(\Lambda)_0$. As mentioned above, Λ is said to be a *noetherian algebra* if $\Lambda = \Lambda_0$, or a *locally artinian algebra* if k is artinian. Furthermore, we assume throughout that our algebras have no semisimple blocks. In this section, we shall review some basic facts about these algebras, including the relevant results from [20].

When k is a field, a large class of examples of locally artinian graded k-algebras is given by path algebras of quivers with relations. If $\Lambda_0 \cong k^n$ and Λ is generated as a k-algebra by $\Lambda_0 \oplus \Lambda_1$, then Λ is realizable as the path algebra of some finite quiver Q, modulo an ideal I of relations that is homogeneous with respect to the natural path-length grading of the path algebra kQ. Other examples of locally artinian graded algebras can be obtained from quivers by assigning arbitrary nonnegative integer degrees (or lengths) to the arrows and factoring out an ideal that is homogeneous with respect to the induced grading on the path algebra, so long as the resulting degree-0 part Λ_0 has finite dimension over k. It is interesting to note that generalized graded quiver algebras can be defined and are still locally artinian for quivers Q with infinitely many arrows between finitely many vertices, provided that only a finite number of arrows have degree n for each positive integer n. In addition, the Yoneda algebras $\mathrm{Ext}^*_A(M, M)$ of finitely generated modules M over an artin algebra A provide many other examples of locally artinian graded algebras.

We let Gr_Λ denote the category of $\mathbb{Z}$-graded right Λ-modules and degree-0 morphisms. By default, we will work with right modules, and all modules (over graded rings) are assumed to be $\mathbb{Z}$-graded unless explicitly stated to the contrary. For a graded module $M = \oplus_i M_i$, we write $M[n]$ for the n^{th} *shift* of M, defined by $M[n]_i = M_{i-n}$ for all $i \in \mathbb{Z}$. We let S_Λ denote the grading shift functor $S : M \mapsto M[1]$, which is a self-equivalence of Gr_Λ. Furthermore, common homological functors such as $\mathrm{Hom}_\Lambda(-,-)$, $\mathrm{Ext}^*_\Lambda(-,-)$, etc. will always refer to Gr_Λ, and thus are to be computed using degree-0 morphisms only. We will denote the set of morphisms of degree d by $\mathrm{Hom}_\Lambda(X,Y)_d \cong \mathrm{Hom}_\Lambda(X[d], Y)$, and similarly for extensions of degree d. We refer the reader to [20] for a detailed treatment of basic homological results in this setting.

For the most part, we will be working inside the full subcategory gr_Λ of finitely generated modules. We will also write $\mathrm{f.p.gr}_\Lambda$ and $\mathrm{f.l.gr}_\Lambda$ for the full subcategories of finitely presented modules and finite length modules, respectively. Notice that gr_Λ is a subcategory of $\mathrm{l.f.gr}_\Lambda$, which consists of the *locally finite* graded modules, meaning those modules $M = \oplus_i M_i$ with each M_i finitely generated over k. As k is noetherian, it is easy to see that this is an abelian category, whereas gr_Λ and $\mathrm{f.p.gr}_\Lambda$ may not be. Furthermore, we append the superscripts $+, -, b$ to the names of these categories to specify the corresponding full subcategories of graded modules that are respectively bounded below, above, or both.

We define stable categories of graded modules as usual and denote them by underlining. Thus $\underline{\mathrm{gr}}_\Lambda$ is the quotient category of gr_Λ obtained by factoring out those morphisms that factor through a projective. If X is a f.g. Λ-module we will sometimes write $\underline{X}$ to indicate that we are viewing X as an object of the stable category. We will also use this notation to denote the largest direct summand of X with no projective summands, which is unique up to isomorphism (see below). If $f \in \mathrm{Hom}_\Lambda(X,Y)$, we write $\underline{f} \in \underline{\mathrm{Hom}}_\Lambda(X,Y)$ for its image in the stable category. Furthermore, we will often abbreviate these Hom-groups as (X,Y) and $(\underline{X},\underline{Y})$ respectively.

We shall say that two graded algebras Λ and Γ are **graded stably equivalent** if there is an equivalence $\alpha : \underline{\mathrm{gr}}_\Lambda \xrightarrow{\approx} \underline{\mathrm{gr}}_\Gamma$ that commutes with the grading shifts in the sense that there is an isomorphism of functors $\eta : \alpha \circ S_\Lambda \xrightarrow{\cong} S_\Gamma \circ \alpha$. In this case it automatically follows that the inverse of α also commutes with the grading shifts. While this restriction appears rather strong, it is reasonable in light of the definition of graded (Morita) equivalence in [10]. It also allows us to recover classical results for artin algebras, and new results for noetherian algebras, as a special case since we may regard such an algebra as a locally finite graded k-algebra that is concentrated in degree 0. In fact, for this trivial grading we have an equivalence $\underline{\mathrm{gr}}_\Lambda \approx \underline{\mathrm{mod}}(\Lambda)^{(\mathbb{Z})}$ from which it follows that the notions of graded and ordinary stable equivalence coincide for algebras concentrated in degree 0. We also point out that, despite the requirement that a graded stable equivalence commutes with the grading shift, several of our results remain true under the weaker hypothesis that $\alpha(M[1]) \cong (\alpha M)[1]$ for all M_Λ, and it would be an interesting problem to determine whether this condition is even truly necessary.

We now state some elementary facts about locally finite graded algebras that follow easily from analogous results for noetherian algebras as found in [15]. First, notice that if $M = \oplus_{i\in\mathbb{Z}} M_i$ is f.g., then $M_i = 0$ for i sufficiently small, and each M_i is f.g. over k. Furthermore, for any f.g. graded Λ-modules M and N, the k-module $\mathrm{Hom}_\Lambda(M,N)$ of degree-0 morphisms is f.g. To see this, suppose M is generated in degrees $d_1,\ldots,d_n$, and observe that the restrictions to the degree-d_i components induce an injection $\mathrm{Hom}_\Lambda(M,N) \hookrightarrow \oplus_{i=1}^n \mathrm{Hom}_k(M_{d_i}, N_{d_i})$. The latter is clearly f.g. over k since each M_{d_i} and N_{d_i} is, and thus so is the former as k is noetherian.

Proposition 1.1. *Let Λ be a locally finite k-algebra, and M a f.g. graded Λ-module. Then*

(a) *M has a decomposition $M = \oplus_{j=1}^n M_j$ into indecomposable modules M_j.*

(b) *$\mathrm{End}_\Lambda(M)$ is a noetherian algebra, and hence is local if M is indecomposable.*

Consequently, gr_Λ is a Krull-Schmidt category.

We let $J_\Lambda = \mathrm{rad}\ \Lambda_0 \oplus \Lambda_{\geq 1}$ denote the *graded Jacobson radical* of Λ. We say that an epimorphism $\pi : P \to M$ with P graded projective is a *(graded) projective cover* if $\ker\ \pi \subseteq PJ$. As we will be working primarily with graded modules, we shall often omit the word "graded" here.

Proposition 1.2. *A locally finite graded algebra Λ is graded semiperfect, meaning that all f.g. graded Λ-modules have graded projective covers. Consequently, all indecomposable graded projective Λ-modules are finitely generated.*

Lemma 1.3 (Nakayama's lemma)**.** *Let Λ be a locally finite graded algebra, and M a graded Λ-module in $\mathrm{l.f.gr}_\Lambda^+$. If $MJ = M$, then $M = 0$. In particular, if $N + MJ = M$, then $N = M$.*

Since k is assumed to be $\mathfrak{m}$-adically complete, so is any finitely generated k-module. Using this fact, we now extend Proposition 2.2 from [20] to show that finitely generated graded modules over a locally finite algebra are J-adically complete.

Lemma 1.4. *Let Λ be a locally finite graded algebra, and let M be a module in $\mathrm{l.f.gr}_\Lambda^+$. Then the natural map $M \to \varprojlim M/MJ^n$ is an isomorphism (where the inverse limit is computed in Gr_Λ).*

Let $E = E(k/\mathfrak{m})$ be the injective envelope of $k/\mathfrak{m}$ over k. Then if R is a noetherian k-algebra, the functor $\mathrm{Hom}_k(-,E)$ gives a duality between the categories of noetherian right R-modules and artinian left R-modules. We can extend this duality to the categories $\mathrm{l.f.gr}_\Lambda$ of locally finite (equivalently, locally noetherian) graded right Λ-modules and $\mathrm{l.a.gr}_{\Lambda^{\mathrm{op}}}$ of *locally artinian* graded left Λ-modules, by which we mean those graded modules with each homogeneous component artinian over k. This duality is given by $D(M)_i = \mathrm{Hom}_k(M_{-i}, E)$ for each $i \in \mathbb{Z}$. In addition, D restricts to dualities $\mathrm{l.f.gr}_\Lambda^+ \to \mathrm{l.a.gr}_{\Lambda^{\mathrm{op}}}^-$,

$\mathrm{gr}_\Lambda \to \mathrm{f.cg.Gr}_{\Lambda^{\mathrm{op}}}$, $\mathrm{f.p.gr}_\Lambda \to \mathrm{f.cp.Gr}_{\Lambda^{\mathrm{op}}}$, and finally $\mathrm{f.l.gr}_\Lambda \to \mathrm{f.l.gr}_{\Lambda^{\mathrm{op}}}$, where $\mathrm{f.cg.Gr}_{\Lambda^{\mathrm{op}}}$ and $\mathrm{f.cp.Gr}_{\Lambda^{\mathrm{op}}}$ denote the categories of finitely cogenerated modules and of finitely copresented modules, respectively. Notice that a f.g. Λ-module M has each M_i f.g., and thus noetherian, over k, while for a f.cg. Λ-module N, each N_i is f.cg., and thus artinian, over k. As a consequence of this duality, we see that the locally finite indecomposable injective modules are the duals of the locally finite indecomposable projectives. As the latter are necessarily f.g. (as Λ is semiperfect, all projectives are direct sums of f.g. indecomposable projectives), the former must be finitely cogenerated. In particular, as long as Λ is finitely generated, any f.g. injective module will have finite length.

The Auslander-Reiten transpose also extends to this context [20], yielding another duality between stable categories

$$\mathrm{Tr} : \underline{\mathrm{f.p.gr}}_\Lambda \to \underline{\mathrm{f.p.gr}}_{\Lambda^{\mathrm{op}}}.$$

Furthermore, with only minor modifications, the proof of existence of almost split sequences given in [20] for locally artinian algebras, extends to this setting to show that Gr_Λ has almost split sequences in the following sense. If $M \in \mathrm{f.p.gr}_\Lambda$ is nonprojective and indecomposable, then there exists an almost split sequence $0 \to D\mathrm{Tr}M \longrightarrow E \longrightarrow M \to 0$ in Gr_Λ [2, 20]. Here, $D\mathrm{Tr}M$ is finitely copresented, while E is usually not finitely generated. However, E is necessarily f.g. if $D\mathrm{Tr}M$ has finite length, and in this case the sequence is also almost split in the smaller category gr_Λ. Dually, there exists an almost split sequence beginning in any finitely copresented, noninjective indecomposable module. In case $\Lambda = \Lambda_0$, we may in fact replace Gr_Λ and gr_Λ with $\mathrm{Mod}(\Lambda)$ and $\mathrm{mod}(\Lambda)$, respectively.

The fact that gr_Λ is a skeletally small, exact Krull-Schmidt category is essentially all that is needed to generalize results on functor categories that Auslander and Reiten used to study stable equivalences between artin algebras in [3, 4]. We will now review the elements of their methods that carry over to this more general setting. Thus let $\mathcal{A}$ be a skeletally small, exact Krull-Schmidt category with enough projectives, and let $\mathrm{mod}(\mathcal{A})$ denote the category of finitely presented contravariant additive functors from $\mathcal{A}$ to the category of abelian groups. Notice that $\mathrm{mod}(\mathcal{A})$ is an exact subcategory of the abelian category $\mathrm{Mod}(\mathcal{A})$ of all contravariant additive functors from $\mathcal{A}$ to the category of abelian groups. Furthermore, let $\underline{\mathrm{mod}}(\mathcal{A})$ denote the full subcategory of $\mathrm{mod}(\mathcal{A})$ consisting of those functors F that vanish on all projective objects of $\mathcal{A}$. This category can be naturally identified with the category $\mathrm{mod}(\underline{\mathcal{A}})$ of finitely presented contravariant additive functors on the stable category $\underline{\mathcal{A}}$, and it follows that the projective objects of $\underline{\mathrm{mod}}(\mathcal{A})$ are the representable functors $(-, \underline{C})$ for objects C of $\mathcal{A}$. Futhermore, the inclusion functor $\underline{\mathrm{mod}}(\mathcal{A}) \to \mathrm{mod}\,\mathcal{A}$ has a left adjoint, which we express as $F \mapsto \underline{F}$ [3]. Of course, if $F \in \underline{\mathrm{mod}}(\mathcal{A})$ then $F \cong \underline{F}$. The following theorem summarizes the results on minimal projective presentations that we shall use. The proof is virtually identical to the one given in [3], and so we omit it.

Theorem 1.5. *Any $F \in \mathrm{mod}(\mathcal{A})$ has a minimal projective presentation $(-, B) \xrightarrow{(-,f)} (-, C) \longrightarrow F \to 0$ in $\mathrm{mod}(\mathcal{A})$. Furthermore, the induced exact sequence $(-, \underline{B}) \xrightarrow{(-,\underline{f})} (-, \underline{C}) \longrightarrow \underline{F} \to 0$ is a minimal projective presentation for $\underline{F}$ in $\underline{\mathrm{mod}}(\mathcal{A})$.*

Thus, if $F \in \underline{\mathrm{mod}}(\mathcal{A})$, then F has a minimal projective presentation $(-, B) \xrightarrow{(-,f)} (-, C) \longrightarrow F \to 0$ in $\mathrm{mod}(\mathcal{A})$ with $f : B \to C$ an epimorphism in $\mathcal{A}$, and this induces a minimal projective presentation $(-, \underline{B}) \xrightarrow{(-,\underline{f})} (-, \underline{C}) \longrightarrow F \to 0$ in $\underline{\mathrm{mod}}(\mathcal{A})$. If f has a kernel A in $\mathcal{A}$, and A, B and C each have partial projective resolutions of length n in $\mathcal{A}$, then as in [3], a long exact sequence of homology groups from [9] provides a partial projective resolution for F in $\underline{\mathrm{mod}}(\mathcal{A})$:

$$(-, \underline{\Omega^n A}) \to (-, \underline{\Omega^n B}) \to (-, \underline{\Omega^n C}) \to \cdots \to (-, \underline{\Omega C}) \to (-, \underline{A}) \to (-, \underline{B}) \to (-, \underline{C}) \to F \to 0.$$

While this resolution is not necessarily minimal beyond the first two terms, its minimality at the third term in certain cases serves as a key ingredient in some of our proofs.

2. Separation of nodes

For path algebras of quivers with relations, a node corresponds to a vertex v of the quiver such that all paths that pass through v are contained in the ideal of relations. This can be thought of as a "local" radical-square-zero condition at v. If one separates such a vertex v into two new vertices, with one a sink and the other a source, the resulting path algebra with relations is stably equivalent to the original one. While this has long been known for artin algebras, these ideas have recently been extended to categories of graded modules over arbitrary graded quiver algebras [21]. The same proofs easily adapt to the case of a locally finite graded algebra. In this section, we review the relevant definitions and results, phrasing them in this generality. Even though we will frequently take advantage of these constructions to assume that our algebras have no nodes, we remark that they so far provide one of the more interesting classes of examples of nontrivial graded stable equivalences between nonartinian graded algebras.

Definition 2.1. *A* **node** *of* Λ *is a nonprojective, noninjective simple module* S_Λ *such that every morphism* $f : S \to M$ *either factors through a projective module or is a split monomorphism.*

Lemma 2.2. *If* S_Λ *is a node and* α *is a stable equivalence, then* αS *is either a node or a simple injective.*

Proof. For any Y_Γ that does not contain αS as a direct summand, we have $\underline{\mathrm{Hom}}_\Gamma(\alpha S, Y) \cong \underline{\mathrm{Hom}}_\Lambda(S, \alpha^{-1}Y) = 0$. Since epimorphisms between indecomposable nonprojective modules do not factor through projectives, we can conclude that αS must be simple. Since αS is not projective, it will be a node unless it is injective. □

As in the artinian case [17], we have the following equivalent characterizations of nodes.

Proposition 2.3 (cf. [17])**.** *Let* S_Λ *be a simple module with projective cover* Q*. Then the following are equivalent.*

(i) *S is projective, injective or a node.*
(ii) *If $f : Q \to P$ is a nonisomorphism with P indecomposable projective, then $f(Q) \subseteq \mathrm{soc}\, P$.*
(iii) *The image of any map $f : Q[i] \to J_\Lambda$ is contained in* $\mathrm{soc}\,\Lambda$ *(this is analogous to saying that S does not occur as a composition factor of $J_\Lambda/\mathrm{soc}\,\Lambda$.)*
(iv) *For all nonisomorphisms $f : P_1 \to Q$ and $g : Q \to P_2$ with P_1, P_2 indecomposable projectives, we have $gf = 0$.*

If S is finitely copresented, then the above conditions are also equivalent to

(v) *Either S is injective or there is a left almost split morphism $\varphi : S \to P$ with P projective.*

We now briefly review the process of separation of nodes. Let $S = S_1 \oplus \cdots \oplus S_n$ be a sum of nonisomorphic nodes (each concentrated in degree 0), let $\mathfrak{a} = \tau_\Lambda(S)$ be the trace ideal of S in Λ, i.e., the ideal generated by the images of all homomorphisms $S[i] \to \Lambda$, and let $\mathfrak{b} = \mathrm{ann}_r(\mathfrak{a})$ be the right annihilator of $\mathfrak{a}$ in Λ. Notice that both of these are homogeneous ideals. Define Γ to be the triangular matrix ring with the given grading

$$\Gamma = \begin{pmatrix} \Lambda/\mathfrak{a} & \mathfrak{a} \\ 0 & \Lambda/\mathfrak{b} \end{pmatrix} = \bigoplus_{i \geq 0} \begin{pmatrix} (\Lambda/\mathfrak{a})_i & \mathfrak{a}_i \\ 0 & (\Lambda/\mathfrak{b})_i \end{pmatrix}.$$

As k is noetherian and $\mathfrak{a}_i \subseteq \Lambda_i$, each $\mathfrak{a}_i$ is f.g. over k, and thus Γ is also locally finite.

Right Γ-modules can be identified with triples (A, B, f) where A is a $\Lambda/\mathfrak{a}$ -module, B is a $\Lambda/\mathfrak{b}$ -module, and $f : A \otimes_{\Lambda/\mathfrak{a}} \mathfrak{a} \to B$ is a morphism of graded $\Lambda/\mathfrak{b}$-modules. We have a functor $F : \mathrm{gr}_\Lambda \to \mathrm{gr}_\Gamma$ defined on Λ-modules X by $F(X) = (X/X\mathfrak{a}, X\mathfrak{a}, \mu)$ where $\mu : X/X\mathfrak{a} \otimes_{\Lambda/\mathfrak{a}} \mathfrak{a} \to X\mathfrak{a}$ is induced by multiplication, and defined on morphisms in the obvious manner. Also notice that F preserves the lengths of finite length modules. The following theorem summarizes the necessary results from [17, 21].

Theorem 2.4 (cf. [17, 21])**.** *Let* Λ*,* S_Λ *and* Γ *be as above. Then* $F : \mathrm{gr}_\Lambda \to \mathrm{gr}_\Gamma$ *commutes with the grading shift and induces an equivalence* $\underline{\mathrm{gr}}_\Lambda \xrightarrow{\approx} \underline{\mathrm{gr}}_\Gamma$*. Furthermore, the nodes of* Γ *are precisely the* Γ*-modules of the form* $(T, 0, 0)$ *where* T *is a node of* Λ *not isomorphic to any* S_i*.*

3. Right noetherian algebras

We now begin our analysis of stable equivalence between graded algebras by showing that if Λ is right noetherian, then so is any graded stably equivalent algebra. We remark that to show a graded algebra is right noetherian, it suffices to check the ascending chain condition on homogeneous right ideals [23]. This, in turn, is equivalent to the condition that any graded submodule of a f.g. projective is also f.g., or in other words that the syzygy of any f.g. graded module is also f.g. Our proof rests on a careful study of the effect of a stable equivalence on a special class of extensions which we now define.

Definition 3.1. *We say that an extension* $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *in* $\mathrm{Ext}^1_\Lambda(C, A)$ *is* **stable** *if* $\underline{f} \neq 0$ *and* **unstable** *if* $\underline{f} = 0$. *It is easy to see that these notions depend only on the equivalence class of* ξ *in* $\mathrm{Ext}^1_\Lambda(C, A)$.

Lemma 3.2. *Let* ξ *denote an extension* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *in* gr_Λ *where* A *has no projective summands , and let* $u : \Omega C \to A$ *denote the connecting morphism. Then* ξ *is unstable if and only if* u *is a split epimorphism.*

Proof. We may realize ξ as the pushout of the short exact sequence $0 \to \Omega C \oplus P_A \xrightarrow{\left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right)} P_C \oplus P_A \xrightarrow{(\pi_C, 0)} C \to 0$ along the epimorphism $(u, \pi_A) : \Omega C \oplus P_A \to A$, where we write $\pi_A : P_A \to A$ and $\pi_C : P_C \to C$ for the projective covers of A and C respectively. We thus have a commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \oplus P_A \arrow[r, "\left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "(u, \pi_A)"'] & P_C \oplus P_A \arrow[r, "(\pi_C, 0)"] \arrow[d, "(p, f\pi_A)"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

If $uv = 1_A$, we have $f = fuv = piv$, which shows $\underline{f} = 0$. Conversely if $\underline{f} = 0$, then f factors through the epimorphism $(p, f\pi_A)$, say via a map $h : A \to P_C \oplus P_A$. Then $\pi_C h = g(p, f\pi_A)h = gf = 0$ implies that $h = \left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} v \\ q \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} v \\ q \end{smallmatrix}\right) : A \to \Omega C \oplus P_A$, and it follows that $f = f(uv + \pi_A q)$. Since f is a monomorphism, we can cancel it to get $1_A = uv + \pi_A q$. As A has no projective summands, $\pi_A q$ is contained in the radical of $\mathrm{End}_\Lambda(A)$, and $uv = 1_A - \pi_A q$ is an automorphism of A. It follows that u splits. $\square$

Proposition 3.3. *The following are equivalent for a finitely generated nonprojective graded module* C *over a graded algebra* Λ.

1. ΩC *is finitely generated.*
2. *For all finitely generated modules* A_Λ, $\mathrm{Ext}^1_\Lambda(C, A[n]) = 0$ *for* $n >> 0$.
3. *For all finitely generated nonprojective modules* A_Λ, $\mathrm{Ext}^1_\Lambda(C, A[n])$ *contains no nonzero stable extensions for* $n >> 0$.

In particular, Λ *is right noetherian if and only if one of these equivalent conditions holds for all f.g. nonprojective* C_Λ.

Proof. If ΩC is finitely generated, then for any finitely generated A_Λ, $\mathrm{Hom}_\Lambda(\Omega C, A[n]) = 0$ for $n >> 0$. Since $\mathrm{Ext}^1_\Lambda(C, A[n])$ is a quotient of $\mathrm{Hom}_\Lambda(\Omega C, A[n])$, we have (1) $\Rightarrow$ (2), and (2) $\Rightarrow$ (3) is trivial. In order to prove (3) $\Rightarrow$ (1), assume that ΩC is not finitely generated. We may of course assume that C has no projective summands. Since ΩC is a submodule of a finitely generated projective, it is locally finite and bounded below, and we have $(\Omega C)J_\Lambda \neq \Omega C$. By Nakayama's lemma, $\Omega C/\Omega C J_\Lambda$ cannot be finitely generated over k, and thus there exists some graded simple S_Λ such that $S[n] | (\Omega C/\Omega C J_\Lambda)$ for infinitely many $n > 0$. The induced epimorphisms $\pi_n : \Omega C \to S[n]$ do not factor through the inclusion $\Omega C \hookrightarrow P_C$, since the image of ΩC is contained in the radical of P_C. Thus the pushout of $0 \to \Omega C \longrightarrow P_C \longrightarrow C \to 0$ along π_n is a nonsplit extension of C by $S[n]$. Moreover, this extension is stable if and only if π_n is not a split epimorphism.

Thus it only remains to consider the case where π_n splits for almost all n. Here, $S[n] | \Omega C$ for infinitely many $n > 0$. We first show that there exists a nonzero nonsplit morphism $u : S \to A$ for some f.g. nonprojective module A_Λ. Such a map can be constructed by choosing two values of n (say, n_0 and n_1) for which $S[n] | \Omega C$, and then taking the appropriate shift of the composite

$$S[n_0] \hookrightarrow \Omega C \hookrightarrow P_C \to P_C/S[n_1] = A[n_0].$$

Clearly, A is nonprojective since $S[n_1] \subset \Omega C \subset P_C J_\Lambda$. Since A and P_C are f.g., $\mathrm{Hom}_\Lambda(P_C, A[n]) = 0$ for $n >> 0$. Thus the maps $u[n]\pi_n : \Omega C \to A[n]$ do not factor through P_C for $n >> 0$. It follows that the pushouts of $0 \to \Omega C \longrightarrow P_C \longrightarrow C \to 0$ along the maps $u[n]\pi_n$ are nonsplit stable extensions of C by $A[n]$ for all sufficiently large values of n. $\square$

Theorem 3.4. *Let $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ be an equivalence that commutes with the grading shift. Then, for any f.g. nonprojective C_Λ, ΩC is f.g. if and only if $\Omega\alpha C$ is f.g. In particular, Γ is right noetherian if and only if Λ is.*

Proof. We assume that ΩC is not f.g., so that by Proposition 3.3 there exists a f.g. nonprojective A_Λ such that $\mathrm{Ext}^1_\Lambda(C, A[n])$ contains nonzero stable extensions for infinitely many $n > 0$. Let us also assume that $\Omega\alpha C$ is f.g. In order to obtain a contradiction, it suffices to show that the existence of a nonzero stable extension in $\mathrm{Ext}^1_\Lambda(C, A)$ for A, C f.g. nonprojective implies the existence of a nonsplit extension in $\mathrm{Ext}^1_\Gamma(\alpha C, \alpha A)$. Let $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a nonsplit extension with $\underline{f} \neq 0$. We may assume that A and C are indecomposable. We thus have the start of a minimal projective resolution for $F \in \mathrm{mod}(\underline{\mathrm{gr}}_\Lambda)$

$$(-, \underline{A}) \xrightarrow{(-,\underline{f})} (-, \underline{B}) \xrightarrow{(-,\underline{g})} (-, \underline{C}) \longrightarrow F \to 0.$$

Applying the equivalence $\bar{\alpha}$ of functor categories induced by α, we get the first three terms of a minimal projective resolution for $\bar{\alpha}F$ in $\mathrm{mod}(\underline{\mathrm{gr}}_\Gamma)$

$$(-, \underline{\alpha A}) \xrightarrow{(-,\alpha\underline{f})} (-, \underline{\alpha B}) \xrightarrow{(-,\alpha\underline{g})} (-, \underline{\alpha C}) \longrightarrow \bar{\alpha}F \to 0.$$

Inside $\mathrm{mod}(\mathrm{gr}_\Gamma)$, $\bar{\alpha}F$ has a minimal projective presentation $(-, \alpha B \oplus P_0) \xrightarrow{(-,v)} (-, \alpha C) \longrightarrow \bar{\alpha}F \to 0$ with $\underline{v} = \alpha\underline{g}$. Regarding $0 \to K \longrightarrow \alpha B \oplus P_0 \xrightarrow{v} \alpha C \to 0$ as a pushout of the projective cover of αC yields the short exact sequence $0 \to \Omega\alpha C \longrightarrow P_{\alpha C} \oplus K \longrightarrow \alpha B \oplus P_0 \to 0$, from which we see that K must be f.g. We therefore obtain the start of a projective resolution for $\bar{\alpha}F$ in $\mathrm{mod}(\underline{\mathrm{gr}}_\Gamma)$

$$(-, \underline{K}) \xrightarrow{(-,\underline{u})} (-, \underline{\alpha B}) \xrightarrow{(-,\underline{v})} (-, \underline{\alpha C}) \longrightarrow \bar{\alpha}F \to 0.$$

Comparing these two projective resolutions for $\bar{\alpha}F$, we see that there must be a split epimorphism $\pi : K \to \alpha A$ with splitting i such that $\underline{ui} = \alpha(\underline{f})$. We now form the pushout

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "u"] \arrow[d, "\pi"] & \alpha B \oplus P_0 \arrow[r, "v"] \arrow[d] & \alpha C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \alpha A \arrow[r] & D \arrow[r] & \alpha C \arrow[r] & 0.
\end{tikzcd}$$

If the bottom sequence splits, π must factor through u, say $\pi = hu$. Thus $\underline{1}_{\alpha A} = \underline{\pi i} = \underline{hui} = \underline{h}\alpha(\underline{f})$. It follows that $\alpha(\underline{f})$ splits, but this contradicts the fact that f does not split. Hence we have produced a nonzero element in $\mathrm{Ext}^1_\Gamma(\alpha C, \alpha A)$. The final statement is now immediate. $\square$

Remarks. (1) Note that the above proof only requires the weaker assumption that α commutes with the grading shift on isomorphism classes of modules. That is, $\alpha(M[1]) \cong (\alpha M)[1]$ for all nonprojective M. It would be interesting to know if this condition is even necessary.

(2) More generally, suppose $\mathcal{A}$ and $\mathcal{B}$ are exact Krull-Schmidt categories with enough projectives that are stably equivalent. Along the lines of the above theorem, one can also ask whether $\mathcal{A}$ being abelian implies that $\mathcal{B}$ is also abelian?

Corollary 3.5. *Let $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ be an equivalence that commutes with the grading shift. If C_Λ is f.p. and nonprojective, then so is αC. Thus α induces an equivalence between the stable categories of f.p. graded modules over Λ and Γ.*

4. Stable extensions

In the proof of Theorem 3.4, we showed that a graded stable equivalence α associates a nonsplit extension in gr_Γ to any stable extension in gr_Λ. In this section we begin a more careful study of the effect of a graded stable equivalence on short exact sequences. Our results will generalize classical results for finite dimensional algebras from [4, 18], but our arguments are necessarily more involved due to the absence of a duality between gr_Λ and $\mathrm{gr}_{\Lambda^{\mathrm{op}}}$. Our first main result in this direction is the following.

Theorem 4.1. *Suppose $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ is an equivalence, and $0 \to A \xrightarrow{f} B \oplus P \xrightarrow{g} C \to 0$ is an exact sequence in gr_Λ, where A, B, C have no projective summands, $\underline{f}$ is right minimal, $\underline{g} \neq 0$, P is projective, and C is f.p. Then there exists an exact sequence $0 \to \alpha A \xrightarrow{u} \alpha B \oplus Q \xrightarrow{v} \alpha C \to 0$ where Q is projective and $\underline{u} = \alpha(\underline{f})$ and $\underline{v} = \alpha(\underline{g})$.*

First, we point out that the term *right minimal* used here has exactly the same meaning as in [6]. In fact, we will now show how this notion can be extended to any skeletally small Krull-Schmidt category $\mathcal{A}$. To do so, we will pass to the functor category $\mathrm{Mod}(\mathcal{A})$ of contravariant additive functors from $\mathcal{A}$ to the category of abelian groups. As in [6], we define a morphism $f : A \to B$ in $\mathcal{A}$ to be *right minimal* if $fs = f$ for some $s \in \mathrm{End}_\mathcal{A}(A)$ implies that s is an automorphism. Clearly, as we have a full and faithful functor $\mathcal{A} \to \mathrm{Mod}(\mathcal{A})$ given by $A \mapsto (-, A)$, we see that $f : A \to B$ is right minimal if and only if $(-, f) : (-, A) \to (-, B)$ is.

Proposition 4.2. *Let $\mathcal{A}$ be a Krull-Schmidt category. For any morphism $f : A \to B$ in $\mathcal{A}$ the following are equivalent.*

(1) *f is right minimal.*
(2) *$(-, f)$ is the projective cover of its image in $\mathrm{Mod}(\mathcal{A})$.*
(3) *For any (nonzero) split monomorphism $g : A' \to A$, we have $fg \neq 0$.*

Moreover, any morphism $f : A \to B$ in $\mathcal{A}$ has a decomposition (unique up to isomorphism) $f = (f', 0) : A' \oplus A'' \to B$ with $f' : A' \to B$ right minimal. We say that $f' : A' \to B$ is the right minimal version of f.

The proof relies on a modification of a result of Auslander [1] that states that any finitely presented functor in $\mathrm{Mod}(\mathcal{A})$ has a minimal projective presentation.

Proposition 4.3. *For a Krull-Schmidt category $\mathcal{A}$, any finitely generated functor F in $\mathrm{Mod}(\mathcal{A})$ has a projective cover.*

Proof. Suppose $f : (-, A) \to F$ is an epimorphism and let $\Gamma = \mathrm{End}_\mathcal{A}(A)$, which is a semiperfect ring. Then $f(A) : (A, A) \cong \Gamma \to F(A)$ is an epimorphism of right Γ-modules. Since Γ is semiperfect and $\mathrm{add}(A)$ is equivalent to proj-Γ, there exists a decomposition $A = A' \oplus A''$ such that $f(A)|_{(A,A')}$ is a projective cover of $F(A)$, while $f(A)|_{(A,A'')} = 0$. Writing $f = (f', f'') : (-, A') \oplus (-, A'') \to F$, we clearly have $f(A)|_{(A,A')} = f'(A)$ and $f(A)|_{(A,A'')} = f''(A)$. We thus see that $f''(A) = 0$, and consequently $f''(A'') = 0$, which by Yoneda's lemma implies that $f'' = 0$. Now since f is surjective, f' must also be surjective. Finally, we must show that $f' : (-, A') \to F$ is an essential epimorphism. The proof of this fact is identical to that given in the proof of I.4.1 of [3], so we omit it here. □

Proof of Proposition 4.2. (3) ⇒ (2): If $(-, f)$ is not the projective cover of its image, then there is some nonzero direct summand A' of A such that $(-, f)$ vanishes on $(-, A') \subseteq (-, A)$. Then clearly f vanishes on A'.

(2) ⇒ (1): Since $(-, f)$ is a projective cover of its image, it is right minimal, and thus so is f.

(1) ⇒ (3): Suppose that $g : A' \to A$ is a split monomorphism such that $fg = 0$. Then, writing $A = g(A') \oplus A''$, we can define an endomorphism h of A by projection onto A'' with kernel $g(A')$. Since $fg = 0$, $f(1_A - h) = 0$, but h is not an automorphism of A and thus f is not right minimal.

For the last remark, note that for an arbitrary $f : A \to B$ we can obtain the right minimal version f' of f by taking the right minimal version $(-, f')$ of $(-, f) : (-, A) \to (-, B)$, which is just the projective cover of the image of $(-, f)$. □

Remark. (3) Analogously, using the category $\mathrm{Mod}(\mathcal{A}^{\mathrm{op}})$ of covariant additive functors from $\mathcal{A}$ to abelian groups, we can obtain dual results for *left minimal* morphisms. Briefly, $f : A \to B$ is left minimal if and only if $(f, -)$ is a projective cover of its image if and only if $gf \neq 0$ for all split epimorphisms $g : B \to B'$.

In order to prove Theorem 4.1 we will need two lemmas.

Lemma 4.4. *Suppose that A, B and C are nonprojective Λ-modules, C is f.p., and we have a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ with $\underline{f}$ right minimal and $\underline{g} \neq 0$, such that $(-, \underline{A}) \xrightarrow{(-,\underline{f})} (-, \underline{B}) \xrightarrow{(-,\underline{g})} (-, \underline{C})$ is an exact* *sequence of functors. Then there exists an exact sequence*

$$0 \to A \oplus Y \xrightarrow{\begin{pmatrix} f & \rho \\ r_1 & r_2 \end{pmatrix}} B \oplus P \xrightarrow{(g,p)} C \to 0,$$

where $\underline{\rho} = 0$, $p : P \to C$ is a projective cover, and $\underline{Y} | \Omega C$.

Proof. Since $(g,p) : B \oplus P \to C$ is surjective and C is f.p., its kernel K is f.g. and we have an exact sequence $0 \to K \xrightarrow{u} B \oplus P \xrightarrow{(g,p)} C \to 0$. Since $\underline{gf} = 0$, we can factor it through the projective cover $p : P \to C$ to obtain $gf = -pq$ for a map $q : A \to P$. Thus $(g,p) \begin{pmatrix} f \\ q \end{pmatrix} = 0$ and $\begin{pmatrix} f \\ q \end{pmatrix}$ factors through $u : K \to B \oplus P$ via a map $i : A \to K$. Meanwhile, we have a projective resolution of the functor $F = \operatorname{coker}(-, \underline{g})$ in $\underline{\mathrm{mod}}(\mathrm{gr}_\Lambda)$, which we can compare to the given exact sequence of functors as follows:

$$\begin{tikzcd}
(-,\underline{K}) \arrow[r, "(-,\underline{u})"] & (-,\underline{B}) \arrow[r, "(-,\underline{g})"] & (-,\underline{C}) \arrow[r] & F \arrow[r] & 0 . \\
(-,\underline{A}) \arrow[u, "(-,\underline{i})"] \arrow[r, "(-,\underline{f})"] & (-,\underline{B}) \arrow[u, equal] \arrow[r, "(-,\underline{g})"] & (-,\underline{C}) \arrow[u, equal] \arrow[r] & F \arrow[u, equal] \arrow[r] & 0
\end{tikzcd}$$

Since $\underline{f}$ is right minimal, $(-, \underline{f})$ is the projective cover of its image. It follows that $(-, \underline{i})$, and hence i too, is a split monomorphism. Thus $K = i(A) \oplus Y$ and writing $u = \begin{pmatrix} u_1 & \rho \\ u_2 & r \end{pmatrix}$ with respect to this decomposition, we clearly have $\underline{\rho} = 0$. Since $\begin{pmatrix} f \\ q \end{pmatrix} = ui$, we get a short exact sequence $0 \to A \oplus Y \xrightarrow{\begin{pmatrix} f & \rho \\ q & r \end{pmatrix}} B \oplus P \xrightarrow{(g,p)} C \to 0$.

To see that $\underline{Y} | \Omega C$, we take the pushout of the above sequence with respect to the projection $A \oplus Y \to \underline{Y}$:

$$\begin{tikzcd}
0 \arrow[r] & A \oplus Y \arrow[r, "\begin{pmatrix} f & \rho \\ q & r \end{pmatrix}"] \arrow[d] & B \oplus P \arrow[r, "(g,p)"] \arrow[d, "(b,p')"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{Y} \arrow[r, "j"] & W \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

As $j = b\rho + p'r$, we have $\underline{j} = 0$. Since $\underline{Y}$ has no projective summands, Lemma 3.2 implies that it is a direct summand of ΩC. □

Lemma 4.5. *Suppose $0 \to A \xrightarrow{\begin{pmatrix} f \\ q \end{pmatrix}} B \oplus P \xrightarrow{(g,p)} C \to 0$ is exact with P projective and C f.p. Suppose $A = A' \oplus A''$ such that $f' = f|_{A'}$ induces the right minimal version $\underline{f'}$ of $\underline{f}$. Then, in the pushout of the given short exact sequence along the projection $\pi : A \to A'$, the induced map j from A' to the pushout B' has $\underline{j}$ right minimal.*

Proof. Let $h : \Omega C \to A$ be the connecting morphism so that πh is the connecting morphism for the pushout sequence. We thus have exact sequences of projectives $(-, \underline{\Omega C}) \xrightarrow{(-,\underline{h})} (-, \underline{A}) \xrightarrow{(-,\underline{f})} (-, \underline{B})$ and $(-, \underline{\Omega C}) \xrightarrow{(-,\underline{\pi h})} (-, \underline{A'}) \xrightarrow{(-,\underline{j})} (-, \underline{B'})$ in $\underline{\mathrm{mod}}(\mathrm{gr}_\Lambda)$. Thus, to show that $\underline{j}$ is right minimal it suffices to show that $(-, \underline{j})$ is the projective cover of its image, or equivalently that $(-, \underline{\pi h})$ is a radical morphism. However, if this were not the case, then there would be a split epimorphism $s : A' \to A_0$ such that $s\pi h$ splits. If t is a splitting for s, we would then have $\underline{f't} = 0$, contradicting the right minimality of $\underline{f'}$. □

Proof of Theorem 4.1. Choosing any maps $u : \alpha A \to \alpha B$ and $v : \alpha B \to \alpha C$ such that $\underline{u} = \alpha \underline{f}$ and $\underline{v} = \alpha \underline{g}$, the sequence $\alpha A \xrightarrow{u} \alpha B \xrightarrow{v} \alpha C$ satisfies the hypotheses of Lemma 4.4. Thus we obtain a short exact sequence

$$0 \to \alpha A \oplus Y \xrightarrow{\begin{pmatrix} u & \rho \\ q & r \end{pmatrix}} \alpha B \oplus P' \xrightarrow{(v,p')} \alpha C \to 0,$$

where $\underline{\rho} = 0$ and $\underline{Y}|\Omega\alpha C$. We consider the following commutative exact pushout diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & & \\
0 \arrow[r] & \alpha A \oplus Y \arrow[r, "\begin{pmatrix} u & \rho \\ q & r \end{pmatrix}"] \arrow[d, "(1{,}0)"'] & \alpha B \oplus P' \arrow[r, "(v{,}p')"] \arrow[d, "(t{,}p_0)"] & \alpha C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \alpha A \arrow[r, "i"] \arrow[d] & V \arrow[r, "\sigma"] \arrow[d] & \alpha C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Clearly, it suffices to show that t induces an isomorphism between αB and $\underline{V}$. For, if $t_1 : \alpha B \to \underline{V}$ denotes the induced isomorphism, and $V = \underline{V} \oplus Q$, we can replace $\underline{V}$ by αB and the maps i and σ by $\begin{pmatrix} t_1^{-1} & 0 \\ 0 & 1_Q \end{pmatrix} i$ and $\sigma \begin{pmatrix} t_1 & 0 \\ 0 & 1_Q \end{pmatrix}$ respectively. The commutativity of the above diagram then shows that these new maps from αA to αB and from αB to αC differ from u and v by maps that factor through projectives.

As $\sigma t = v$, we have $\underline{\sigma} \neq 0$ and $\underline{t} \neq 0$. Furthermore, $\underline{i}$ is right minimal by Lemma 4.5. Applying α^{-1}, we obtain maps $j : A \to \alpha^{-1}\underline{V}$, $t' : B \to \alpha^{-1}\underline{V}$, and $\tau : \alpha^{-1}\underline{V} \to C$, lifting $\alpha^{-1}(\underline{i}), \alpha^{-1}(\underline{t})$ and $\alpha^{-1}(\underline{\sigma})$ respectively. For ease of reference, we illustrate these maps in the following commutative diagram in the stable category.

$$\begin{tikzcd}
\underline{A} \arrow[r, "\underline{f}"] \arrow[d, equal] & \underline{B} \arrow[r, "\underline{g}"] \arrow[d, "\underline{t'}"] & \underline{C} \arrow[d, equal] \\
\underline{A} \arrow[r, "\underline{j}"] & \alpha^{-1}\underline{V} \arrow[r, "\underline{\tau}"] & \underline{C}
\end{tikzcd}$$

Since we have a short exact sequence $0 \to \alpha A \xrightarrow{i} V \xrightarrow{\sigma} \alpha C \to 0$ with $\underline{i}$ right minimal and $\underline{\sigma} \neq 0$, Lemma 4.4 applies to the sequence $A \xrightarrow{j} \alpha^{-1}V \xrightarrow{\tau} C$, yielding a short exact sequence

$$0 \to A \oplus Y' \xrightarrow{\begin{pmatrix} j & \rho \\ r_1 & r_2 \end{pmatrix}} \alpha^{-1}\underline{V} \oplus P_C \xrightarrow{(\tau,\pi)} C \to 0$$

with $\pi : P_C \to C$ a projective cover and $\underline{\rho} = 0$. As $\underline{\tau t'} = \underline{g}$, we have $g = \tau t' + \pi w$ for a map $w : B \to P_C$, and clearly $p = \pi l$ for some $l : P \to P_C$. This leads to the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\begin{pmatrix} f \\ q \end{pmatrix}"] \arrow[d, "\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}"] & B \oplus P \arrow[r, "(g{,}p)"] \arrow[d, "\begin{pmatrix} t' & 0 \\ w & l \end{pmatrix}"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \oplus Y' \arrow[r, "\begin{pmatrix} j & \rho \\ r_1 & r_2 \end{pmatrix}"'] & \alpha^{-1}\underline{V} \oplus P_C \arrow[r, "(\tau{,}\pi)"'] & C \arrow[r] & 0
\end{tikzcd}$$

We now have $\underline{j} = \underline{t'f} = \underline{js_1}$, and since $\underline{j}$ is right minimal and A has no projective summands, s_1 is an automorphism of A. It now follows that $s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ is a split monomorphism and thus that $\begin{pmatrix} t' & 0 \\ w & l \end{pmatrix}$ is a also monomorphism with cokernel isomorphic to Y'. We have a commutative square

$$\begin{tikzcd}
A \oplus Y' \arrow[r, "\begin{pmatrix} j & \rho \\ r_1 & r_2 \end{pmatrix}"] \arrow[d, "(-s_2 s_1^{-1}, 1)"] & \alpha^{-1}\underline{V} \oplus P_C \arrow[d] \\
Y' \arrow[r, "\cong"] & Y'
\end{tikzcd}$$

which shows that the automorphism of Y' obtained by going down and to the right factors through a projective. Hence Y' must be projective. Thus $\begin{pmatrix} t' & 0 \\ w & l \end{pmatrix}$ splits, and $t' : B \to \alpha^{-1}\underline{V}$ must be an isomorphism. Since $\underline{t} = \alpha \underline{t'}$, t must also induce an isomorphism between αB and $\underline{V}$. $\square$

5. Syzygies and stable Grothendieck groups

Our present goal is to show that a stable equivalence between graded algebras commutes with the syzygy regarded as an operator on isomorphism classes of modules. As in the finite dimensional case, the possibility of nodes occurring as summands of the syzygy prevents this from holding in complete generality. However, we will see that we can always say something about the nonprojective summands of the syzygies that are not nodes. We thus introduce the notation $\underline{\underline{C}}$ to denote the maximal direct summand of C (unique up to isomorphism) containing no projective modules or nodes as direct summands. In order to prove our main result, we will need the following lemma.

Lemma 5.1. *Suppose $f_i : X_i \to Y_i$ $(1 \leq i \leq n)$ are nonzero radical morphisms between indecomposable objects in a Krull-Schmidt category $\mathcal{A}$. Then the map $f = \oplus_{i=1}^n f_i : \oplus_{i=1}^n X_i \to \oplus_{i=1}^n Y_i$ is a left and right minimal radical morphism.*

Proof. If $u_i : Y_i \to Y = \oplus_{j=1}^n Y_j$ and $\pi_i : X = \oplus_{j=1}^n X_j \to X_i$ are the canonical inclusions and projections, we have $f = \sum_{i=1}^n u_i f_i \pi_i$, which belongs to the radical of $\mathcal{A}$ since each f_i does. Now suppose that $f|_W = 0$ where $X = W \oplus Z$. We may assume that W is indecomposable, and thus that it has a local endomorphism ring. Hence, W satisfies the exchange property, and thus there exist direct summands X_j' of X_j for each $1 \leq j \leq n$ such that $X = W \oplus X_1' \oplus \cdots X_n'$. By the uniqueness of the direct sum decomposition of X, some X_i' must be 0. However, letting $v_i : Y \to Y_i$ be the projection, this implies that $v_i f = 0$ and hence that $f_i = v_i f|_{X_i} = 0$, a contradiction. The proof of left minimality is analogous. $\square$

Theorem 5.2. *Let $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ be an equivalence that commutes with the grading shift. If C_Λ is f.p. and indecomposable, then $\underline{\underline{\Omega\alpha C}} \cong \alpha\underline{\underline{\Omega C}}$. In particular, if Λ and Γ have no nodes, we have $\underline{\Omega\alpha C} \cong \alpha\underline{\Omega C}$.*

Proof. Assume that $\underline{\underline{\Omega C}} \neq 0$. For each indecomposable direct summand D_i of $\underline{\underline{\Omega C}}$, there exists a nonisomorphism $f_i : D_i \to A_i$ with A_i indecomposable and $\underline{f_i} \neq 0$. Let $f : \Omega C \to A := \oplus A_i$ be defined by extending $\oplus f_i$ by zero on the remaining summands of ΩC, and form the pushout

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[r] \arrow[d, "f"] & P_C \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "h"] & B \arrow[r, "g"] & C \arrow[r] & 0.
\end{tikzcd}$$

We claim that

$$(-, \underline{\underline{\Omega C}}) \overset{(-,\underline{f})}{\longrightarrow} (-, \underline{A}) \overset{(-,\underline{h})}{\longrightarrow} (-, \underline{B}) \overset{(-,\underline{g})}{\longrightarrow} (-, \underline{C}) \longrightarrow F \to 0$$

is the start of a minimal projective resolution of $F = \mathrm{coker}(-, g)$ in $\underline{\mathrm{mod}}(\mathrm{gr}_\Lambda)$. Clearly, it suffices to check that each of the maps is right minimal. By the above lemma applied to $\underline{f}$ in $\underline{\mathrm{gr}}_\Lambda$, we see that $(-, \underline{f})$ is right minimal. We also see that $(-, \underline{f})$ is a radical map between projectives, and thus it follows that its cokernel is a projective cover. Hence $(-, \underline{h})$ must be right minimal. Finally, we claim that $(-, \underline{h})$ is also a radical morphism, from which it will follow that $(-, \underline{g})$ is right minimal. Indeed, otherwise there would be a map $t : B \to B'$ such that th is a split epimorphism. But since f is left minimal, we have $thf \neq 0$, contradicting $hf = 0$. Applying $\bar{\alpha}$ we get the start of a minimal projective resolution

$$(-, \alpha\underline{\underline{\Omega C}}) \longrightarrow (-, \alpha\underline{A}) \longrightarrow (-, \alpha\underline{B}) \longrightarrow (-, \alpha\underline{C}) \longrightarrow \bar{\alpha}F \to 0.$$

But, as in the proof of Theorem 3.4, $\bar{\alpha}F$ also has a projective resolution

$$(-,\underline{\Omega\alpha C}) \longrightarrow (-,\underline{K}) \longrightarrow (-,\underline{\alpha B}) \longrightarrow (-,\underline{\alpha C}) \longrightarrow \bar{\alpha}F \to 0,$$

for some f.g. module K. Comparing these resolutions yields $\alpha\underline{\underline{\Omega C}}|\underline{\Omega\alpha C}$. In fact, $\alpha\underline{\underline{\Omega C}}$ can have no nodes as summands since $\underline{\underline{\Omega C}}$ has no nodes or simple injectives as summands. Thus, we have $\alpha\underline{\underline{\Omega C}}|\underline{\underline{\Omega\alpha C}}$. Completely analogously we have $\alpha^{-1}\underline{\underline{\Omega\alpha C}}|\underline{\underline{\Omega C}}$, and it follows that $\alpha\underline{\underline{\Omega C}} \cong \underline{\underline{\Omega\alpha C}}$.

If $\underline{\underline{\Omega C}} = 0$, and $\underline{\underline{\Omega\alpha C}} \neq 0$, the above argument shows that the latter is a direct summand of $\alpha\underline{\underline{\Omega C}} = 0$, which is a contradiction. $\square$

Remark. (4) The only place in the above proof where we have used the assumption that α commutes with the grading shift is in the application of Theorem 3.4 to conclude that $\Omega\alpha C$ is f.g. If we assume from the beginning that Λ and Γ are right noetherian, then we have $\underline{\underline{\Omega\alpha C}} \cong \alpha\underline{\underline{\Omega C}}$ even if α does not commute with the grading shift. Likewise, this proof carries over to the case where $\mathcal{A}$ and $\mathcal{B}$ are any stably equivalent abelian Krull-Schmidt categories with enough projectives.

Corollary 5.3. *Let $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ be a stable equivalence (that does not necessarily commute with the grading shift) between two right noetherian algebras without nodes. Then $\mathrm{pd}_\Gamma \alpha C = \mathrm{pd}_\Lambda C$ for all f.g. nonprojective Λ-modules C.*

For a second corollary, we turn our attention to finitely presented graded modules. We say that a graded algebra Λ is *right graded coherent* if every f.g. graded right ideal of Λ is f.p. As in the nongraded setting, this is easily seen to be equivalent to either (1) every f.g. graded submodule of a f.p. graded right Λ-module is f.p.; or (2) the syzygy of any f.p. graded right module is f.p.; or (3) the category $\mathrm{f.p.gr}_\Lambda$ is abelian.

Corollary 5.4. *Assume either that Λ and Γ are f.g. as algebras–so that the simples are f.p.–or that they have no nodes, and let $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ be an equivalence that commutes with the grading shift. Then Λ is right graded coherent if and only if Γ is.*

Proof. If Λ is right graded coherent, ΩC is f.p. whenever C is f.p. Thus, for all C f.p., ΩC and $\Omega^2 C$ are f.g. By Theorem 3.4, $\Omega\alpha C$ and $\Omega\alpha\Omega C$ are f.g. over Γ. We have $\underline{\Omega C} \cong \underline{\underline{\Omega C}} \oplus S$, where S_Λ is a direct sum of nodes. Thus $\alpha\underline{\Omega C} \cong \alpha\underline{\underline{\Omega C}} \oplus T \cong \underline{\underline{\Omega\alpha C}} \oplus T$, where $T_\Gamma = \alpha S$ is a direct sum of nodes and simple injectives. Clearly, T is f.g. since $\alpha\Omega C$ is. Since $\Omega\alpha C$ is also f.g., there exists a finite direct sum T'_Γ of nodes such that $\alpha\underline{\Omega C} \oplus T' \cong \underline{\Omega\alpha C} \oplus T$. Taking the syzygy of each side, and noting that $\Omega\underline{X} \cong \Omega X$, we have

$$\Omega\alpha\Omega C \oplus \Omega T' \cong \Omega^2\alpha C \oplus \Omega T.$$

Since T and T' are f.g. and semisimple, they are f.p., and hence their syzygies are f.g. As $\Omega\alpha\Omega C$ is f.g., so is $\Omega^2\alpha C$. Thus $\Omega\alpha C$ is f.p., and we conclude that Γ is right graded coherent. $\square$

Remarks. (5) The following example shows that the restrictions on the algebras in the above corollary are indeed necessary. Let Λ be the graded quiver algebra of the quiver Q consisting of one vertex and infinitely many loops $\{x_i\}_{i\in\mathbb{N}}$, with x_i in degree i, with the relations $x_i x_j = 0$ for all i and j. The single vertex of Q corresponds to a node of Λ, and separating it yields a stably equivalent algebra Γ that is the path algebra of the quiver consisting of two vertices and infinitely many parallel arrows $\{x'_i\}_{i\in\mathbb{N}}$, with x'_i in degree i.

Notice that Γ is hereditary, so any f.g. right ideal is projective and thus also f.p. On the other hand, the right ideal $x_1\Lambda$ is a simple Λ-module which is clearly not f.p. We thus see that Γ is right graded coherent, while Λ is not.

(6) We do not know whether the above corollary would still hold under the weaker hypothesis of a stable equivalence $\alpha : \underline{\mathrm{f.p.gr}}_\Lambda \to \underline{\mathrm{f.p.gr}}_\Gamma$ between categories of f.p. modules. In essence, this is an interesting special case of the question raised in Remark (2) of Section 3.

As another application of Theorem 5.2, we can strengthen the results from the previous section on short exact sequences. In fact, this will then allow us to show that graded stably equivalent algebras have isomorphic stable Grothendieck groups, generalizing a theorem of the second-named author for stable equivalences between artin algebras [18].

Recall that the **stable Grothendieck group** of a skeletally small abelian Krull-Schmidt category $\mathcal{A}$ with enough projectives is defined as the quotient $K_0^s(\mathcal{A}) = L(\mathcal{A})/R(\mathcal{A})$, where $L(\mathcal{A})$ is the free abelian group on the isomorphism classes $[X]$ of objects of $\mathcal{A}$ and $R(\mathcal{A})$ is the subgroup generated by all elements of the form $[A] - [B] + [C]$ for which there exists an exact sequence $0 \to A \oplus P \xrightarrow{f} B \oplus Q \xrightarrow{g} C \to 0$ with P and Q projective, and where A, B may be 0. Clearly, to show that a graded stable equivalence $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ between right noetherian algebras induces an isomorphism $K_0^s(\mathrm{gr}_\Lambda) \cong K_0^s(\mathrm{gr}_\Gamma)$ it suffices to show that it preserves short exact sequences of the above form. The following theorem takes care of the majority of such sequences.

Theorem 5.5. *Suppose* $0 \to A_1 \oplus A_2 \xrightarrow{\begin{pmatrix} f_1 & f_2 \\ q_1 & q_2 \end{pmatrix}} B \oplus P \xrightarrow{(g,p)} C \to 0$ *is a short exact sequence in* gr_Λ*, where* P *is projective,* A_1 *and* B *have no projective summands, and* B *and* C *are f.p. Further, assume that* $\underline{f_1}$ *is right minimal,* $\underline{f_2} = 0$ *and* $\underline{g} \neq 0$*. Then there exists a short exact sequence* $0 \to \alpha A_1 \oplus Y \xrightarrow{\begin{pmatrix} f_1' & f_2' \\ q_1' & q_2' \end{pmatrix}} \alpha B \oplus P' \xrightarrow{(g',p')} \alpha C \to 0$ *in* gr_Γ*, where* P' *is projective,* $\underline{\underline{Y}} \cong \underline{\underline{\alpha A_2}}$*,* $\underline{f_1'} = \alpha(\underline{f_1})$*,* $\underline{g'} = \alpha(\underline{g})$*, and* $\underline{f_2'} = 0$*.*

Proof. We start by taking the pushout of the given sequence along the projection $A_1 \oplus A_2 \to A_1$ to get the following commutative exact diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A_2 \arrow[r, equal] \arrow[d] & A_2 \arrow[d] & & \\
0 \arrow[r] & A_1 \oplus A_2 \arrow[r, "\begin{pmatrix} f_1 & f_2 \\ q_1 & q_2 \end{pmatrix}"] \arrow[d, "{(1,0)}"'] & B \oplus P \arrow[r, "{(g,p)}"] \arrow[d, "t"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "\begin{pmatrix} i \\ r \end{pmatrix}"] \arrow[d] & B_1 \oplus Q \arrow[r, "{(\sigma,\pi)}"] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Here, Q is projective and B_1 has no projective summands. Clearly $\underline{t}$ and $\underline{\sigma}$ are nonzero. By Lemma 4.5, $\underline{i}$ is right minimal, and thus we may apply Theorem 4.1 to the bottom row to obtain an exact sequence $0 \to \alpha A_1 \xrightarrow{\begin{pmatrix} i' \\ r' \end{pmatrix}} \alpha B_1 \oplus Q' \xrightarrow{(\sigma',\pi')} \alpha C \to 0$ in gr_Γ, where $\underline{i'} = \alpha(\underline{i})$ and $\underline{\sigma'} = \alpha(\underline{\sigma})$. At the same time, Lemma 4.4 applied to the sequence $(-, \alpha\underline{A_1}) \xrightarrow{(-,\alpha\underline{f_1})} (-, \alpha\underline{B}) \xrightarrow{(-,\alpha\underline{g})} (-, \alpha\underline{C})$ yields an exact sequence

$$0 \to \alpha A_1 \oplus Y \xrightarrow{\begin{pmatrix} f_1' & f_2' \\ q_1' & q_2' \end{pmatrix}} \alpha B \oplus P' \xrightarrow{(g',p')} \alpha C \to 0$$

in gr_Γ with p' a projective cover, $\underline{f_1'} = \alpha(\underline{f_1})$, $\underline{f_2'} = 0$ and $\underline{g'} = \alpha(\underline{g})$. Let $t_0 : B \to B_1$ be the appropriate component of t. Since $\underline{\sigma t_0} = \underline{g}$, for any lift t_0' of $\alpha\underline{t_0}$ we have $\underline{\sigma' t_0'} = \alpha(\underline{\sigma t_0}) = \alpha(\underline{g}) = \underline{g'}$. Thus $\sigma' t_0' - g' = p's$

for some $s : \alpha B \to P'$, and we obtain a commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \alpha A_1 \oplus Y \arrow[r, "\begin{pmatrix} f_1' & f_2' \\ q_1' & q_2' \end{pmatrix}"] \arrow[dd, "(s_1,s_2)"'] & \alpha B \oplus P' \arrow[r, "(g',p')"] \arrow[d, "\begin{pmatrix} t_0' & 0 \\ s & 1 \end{pmatrix}"'] & \alpha C \arrow[r] \arrow[d, equal] & 0 \\
& & \alpha B_1 \oplus P' \arrow[r, "(\sigma',p')"] \arrow[d, "\begin{pmatrix} 1 & p_1 \\ 0 & p_2 \end{pmatrix}"'] & \alpha C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \alpha A_1 \arrow[r, "\begin{pmatrix} i' \\ r' \end{pmatrix}"'] & \alpha B_1 \oplus Q' \arrow[r, "(\sigma',\pi')"] & \alpha C \arrow[r] & 0
\end{tikzcd}$$

Here, $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} : P' \to \alpha B_1 \oplus Q'$ exists since P' is projective, and $(s_1, s_2) : \alpha A_1 \oplus Y \to \alpha A_1$ is the map induced on the kernels by the commutativity of the right hand rectangle. We easily see that $\underline{i's_1} = \underline{t_0'f_1'}$, and since $\underline{t_0f_1} = \underline{i}$ we obtain $\underline{i's_1} = \alpha(\underline{t_0f_1}) = \alpha(\underline{i}) = \underline{i'}$. Since we know that $\underline{i}$, and hence $\underline{i'}$, is right minimal, $\underline{s_1}$ must be an isomorphism. Of course, as $\alpha A'$ has no projective summands, this implies that s_1 is an isomorphism. Hence the map (s_1, s_2) is a split epimorphism with kernel isomorphic to Y.

From the middle column of the above diagram, we now obtain an exact sequence $0 \to Y \xrightarrow{\gamma} \alpha B \oplus P' \xrightarrow{\begin{pmatrix} t_0' & * \\ * & * \end{pmatrix}} \alpha B_1 \oplus Q' \to 0$ where the $*$'s all represent maps that factor through projectives. Similarly, in gr_Λ we have a short exact sequence $0 \to A_2 \xrightarrow{\begin{pmatrix} f_2 \\ * \end{pmatrix}} B \oplus P \xrightarrow{\begin{pmatrix} t_0 & * \\ * & * \end{pmatrix}} B_1 \oplus Q \to 0$ with $\underline{f_2} = 0$. The latter yields a projective resolution

$$0 \to (-, \underline{B}) \xrightarrow{(-,\underline{t_0})} (-, \underline{C}) \longrightarrow \mathrm{coker}(-, \underline{t_0}) \to 0.$$

If we apply $\bar{\alpha}$ we get a projective resolution of the same form for the cokernel of $(-, \alpha\underline{t_0}) = (-, \underline{t_0'})$. An isomorphic projective resolution must be induced by the above short exact sequence starting in Y, and thus it follows that $\underline{\gamma} = 0$. We now wish to apply Lemma 3.2 to this sequence. If Y has projective summands we may first form the pushout along the projection $Y \to \underline{Y}$, as the resulting extension is easily seen to be unstable as well. Thus it follows that $\underline{Y}$ is isomorphic to a direct summand of $\underline{\Omega\alpha B_1}$. As we are assuming that B is f.p., we see that αB and αB_1 are as well, and thus the complement to $\underline{Y}$ must be isomorphic to $\underline{\Omega\alpha B}$ by the snake lemma applied to the appropriate analogue of the diagram in the proof of 3.2. Hence we have $\underline{Y} \oplus \underline{\Omega\alpha B} \cong \underline{\Omega\alpha B_1}$ and similarly the other exact sequence yields $\underline{A_2} \oplus \underline{\Omega B} \cong \underline{\Omega B_1}$. Therefore, by Theorem 5.2 $\underline{\underline{\alpha A_2 \oplus \alpha\Omega B}} \cong \underline{\underline{\alpha\Omega B_1}} \cong \underline{\underline{\Omega\alpha B_1}} \cong \underline{\underline{Y \oplus \Omega\alpha B}}$, and it follows that $\underline{\underline{\alpha A_2}} \cong \underline{\underline{Y}}$. □

Theorem 5.6. *Suppose that $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ is an equivalence where Λ and Γ are right noetherian and without nodes. Then $K_0^s(\mathrm{gr}_\Lambda) \cong K_0^s(\mathrm{gr}_\Gamma)$.*

Proof. Let $0 \to A \oplus Q \xrightarrow{f} B \oplus P \xrightarrow{g} C \to 0$ be a short exact sequence in gr_Λ with P, Q projective, and such that A and B have no projective summands. We consider several cases. First, if $\underline{g} = 0$, then we have an exact sequence of projectives $0 \to (-, \underline{\Omega C}) \to (-, \underline{A}) \to (-, \underline{B}) \to 0$ in $\underline{\mathrm{mod}}(\mathrm{gr}_\Lambda)$. This sequence must split, and we obtain $A \cong B \oplus \underline{\Omega C}$. Hence $\alpha A \cong \alpha B \oplus \underline{\Omega\alpha C}$ by Theorem 5.2, and noting that we always have $[\Omega\alpha C] = -[\alpha C]$ in the stable Grothendieck group, we obtain the relation $[\alpha A] + [\alpha C] = [\alpha B]$ for $K_0^s(\mathrm{gr}_\Gamma)$. Similarly, if $\underline{f} = 0$, we can use the exact sequence $0 \to (-, \underline{\Omega B}) \to (-, \underline{\Omega C}) \to (-, \underline{A}) \to 0$ to conclude that $\underline{\Omega C} \cong \underline{\Omega B} \oplus \underline{A}$. Again by Theorem 5.2, applying α shows that the analogous relation holds in $K_0^s(\mathrm{gr}_\Gamma)$. Thus we may now assume that $\underline{f}$ and $\underline{g}$ are nonzero, and apply the previous theorem. □

Remark. (7) Similarly, if Λ and Γ are graded right coherent, then we obtain an isomorphism $K_0^s(\mathrm{f.p.gr}_\Lambda) \cong K_0^s(\mathrm{f.p.gr}_\Gamma)$.

6. Projective extensions

We have already seen in Theorem 4.1 that for C_Λ f.p., a graded stable equivalence $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ associates a nonsplit extension of αC by αA to any nonsplit stable extension of C by A. We now undertake a more in-depth comparison of the extension groups $\mathrm{Ext}^1_\Lambda(C, A)$ and $\mathrm{Ext}^1_\Gamma(\alpha C, \alpha A)$, parallel to the analysis of the last two sections. Assuming that Λ and Γ have no nodes and that C_Λ is f.p., α always induces an isomorphism $(\underline{\Omega C}, \underline{A}) \xrightarrow{\cong} (\underline{\Omega\alpha C}, \underline{\alpha A})$, and these homomorphism groups naturally arise as quotients of the aforementioned Ext-groups. We thus concentrate on the kernels of these quotient maps, that is, on the extensions of A by C corresponding to maps $\Omega C \to A$ that factor through a projective. Much of what follows makes sense in any abelian category $\mathcal{C}$ with enough projectives, and we thus start off in this general setting and later specialize to the category Gr_Λ, and to gr_Λ when C is f.p. While projective covers may not exist in $\mathcal{C}$, we will still write ΩC for the kernel of some fixed epimorphism $\pi_C : P_C \to C$ with P_C projective.

Definition 6.1. *Let $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an extension in $\mathrm{Ext}^1_{\mathcal{C}}(C, A)$ with connecting morphism $h : \Omega C \to A$, which is unique up to the addition of a map that factors through the inclusion $\Omega C \hookrightarrow P_C$. We say that ξ is* **projective** *if $\underline{h} = 0$. Clearly, this definition depends only on the equivalence class of the extension ξ, and is independent of the choices of π_C and h. We let $K(C, A) \subseteq \mathrm{Ext}^1_{\mathcal{C}}(C, A)$ denote the subset of equivalence classes of projective extensions.*

The following lemma implies that $K(-,-)$ is an additive sub-bifunctor of $\mathrm{Ext}^1_{\mathcal{C}}(-,-)$ [7]. The proof is routine and not essential to our discussion, so we omit it.

Lemma 6.2. *The class of projective extensions is closed under taking direct sums and forming pushouts or pullbacks.*

We can now study the relative homology of $\mathcal{C}$ with respect to the sub-bifunctor $K(-,-)$ of $\mathrm{Ext}^1_{\mathcal{C}}(-,-)$ as in [7]. We say that an object C of $\mathcal{C}$ is K-projective if $K(C, A) = 0$ for all objects A in $\mathcal{C}$. Clearly, this is equivalent to all projective extensions $0 \to A \longrightarrow B \longrightarrow C \to 0$ splitting.

Lemma 6.3. *An object C is K-projective if and only if $\mathrm{Ext}^1_{\mathcal{C}}(C, P) = 0$ for all projectives P.*

Proof. The forward direction is clear since any extension $0 \to P \longrightarrow B \longrightarrow C \to 0$ is projective. In fact, $K(C, P) = \mathrm{Ext}^1_{\mathcal{C}}(C, P)$ for all C and all projectives P. Conversely, if $\xi : 0 \to A \longrightarrow B \longrightarrow C \to 0$ is projective with connecting morphism $h : \Omega C \to A$, then h factors through an epimorphim $\pi_A : P_A \to A$ with P_A projective. It follows that ξ arises as the pushout via π_A of an extension in $\mathrm{Ext}^1_{\mathcal{C}}(C, P_A) = 0$. Hence ξ splits. $\square$

If C is a f.g. right Λ-module, we shall say that C is K-projective in gr_Λ if $K(C, A) = 0$ for all f.g. A. The above lemma now reduces to say that C is K-projective if and only if $\mathrm{Ext}^1_\Lambda(C, \Lambda[i]) = 0$ for all $i \in \mathbb{Z}$.

Proposition 6.4. *Let $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ be an equivalence that commutes with the grading shift, and suppose that Λ and Γ have no nodes. Then a f.p. nonprojective indecomposable C_Λ is K-projective if and only if αC is.*

Proof. If C is not K-projective, then according to the preceding lemma, there exists a nonsplit extension $\xi : 0 \to P \longrightarrow B \xrightarrow{g} C \to 0$ for some indecomposable projective P. If B is projective, then ΩC is isomorphic to a summand of P and hence projective. It follows from Theorem 5.2 that $\Omega\alpha C$ is also projective, from which we deduce that αC is not K-projective. We may thus assume that B is not projective. Furthermore, from the short exact sequence $0 \to \Omega C \longrightarrow P_C \oplus P \longrightarrow B \to 0$, we observe that $\underline{\Omega B} \cong \underline{\Omega C}$. Now, we have a minimal projective resolution

$$0 \to (-, \underline{B}) \xrightarrow{(-,\underline{g})} (-, \underline{C}) \longrightarrow F \to 0$$

for the functor $F = \mathrm{coker}(-, g)$ in $\mathrm{mod}(\underline{\mathrm{gr}}_\Lambda)$. Applying $\bar{\alpha}$, we obtain a minimal projective resolution $0 \to (-, \underline{\alpha B}) \xrightarrow{(-,\alpha \underline{g})} (-, \underline{\alpha C}) \longrightarrow \bar{\alpha} F \to 0$ in $\mathrm{mod}(\underline{\mathrm{gr}}_\Gamma)$, and it follows that there is a nonsplit short exact sequence $\zeta : 0 \to A \xrightarrow{u} \alpha B \oplus P_0 \xrightarrow{v} \alpha C \to 0$ with $\underline{v} = \alpha \underline{g}$, which corresponds to the minimal projective resolution of $\bar{\alpha} F$ in $\mathrm{mod}(\mathrm{gr}_\Gamma)$. We know that $A = \ker v$ is f.g. since αC is f.p. and $\alpha B \oplus P_0$ is f.g. This short exact sequence, in turn, gives rise to a long exact sequence of functors on $\underline{\mathrm{gr}}_\Gamma$

$$\cdots \to (-, \underline{\Omega A}) \xrightarrow{0} (-, \underline{\Omega\alpha B}) \longrightarrow (-, \underline{\Omega\alpha C}) \longrightarrow (-, \underline{A}) \xrightarrow{0} (-, \underline{\alpha B}) \longrightarrow (-, \underline{\alpha C}) \longrightarrow \bar{\alpha} F \to 0.$$

We conclude that $\underline{u} = 0$ by comparing this sequence to the minimal projective resolution of $\bar{\alpha}F$ in $\mathrm{mod}(\underline{\mathrm{gr}}_\Gamma)$, and it follows that $\Omega\underline{u} = 0$ as well. Thus, we have a split short exact sequence $0 \to (-, \underline{\Omega\alpha B}) \longrightarrow (-, \underline{\Omega\alpha C}) \longrightarrow (-, \underline{A}) \to 0$. However, $\underline{\Omega\alpha B} \cong \underline{\alpha\Omega B} \cong \underline{\alpha\Omega C} \cong \underline{\Omega\alpha C}$ by Theorem 5.2. Therefore, $\underline{A} = 0$ and we have shown that A_Γ is projective. Consequently, αC is not K-projective. □

Corollary 6.5. *Let $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ be an equivalence that commutes with the grading shift, and suppose that Λ and Γ have no nodes. Let C_Λ be a f.p. nonprojective indecomposable such that $\Omega^i C$ is f.g. for all $i \geq 1$. Then, for each $i \geq 1$, $\mathrm{Ext}^i_\Lambda(C, \Lambda[j]) = 0$ for all $j \in \mathbb{Z}$ if and only if $\mathrm{Ext}^i_\Gamma(\alpha C, \Gamma[j]) = 0$ for all $j \in \mathbb{Z}$.*

Proof. The $i = 1$ case is simply a restatement of the above proposition. For $i > 1$, it follows from the fact that $\mathrm{Ext}^i_\Lambda(C, \Lambda[j]) \cong \mathrm{Ext}^1_\Lambda(\Omega^{i-1}C, \Lambda[j])$ combined with the $i = 1$ case and Theorem 5.2. □

Proposition 6.6. *With the above notation, $K(C, A) = 0$ for f.g. nonprojective indecomposable Λ-modules A and C with C f.p. if and only if $K(\alpha C, \alpha A) = 0$.*

Proof. We shall show that $K(C, A) \neq 0$ implies that $K(\alpha C, \alpha A) \neq 0$. Thus, let $\xi : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a nonsplit projective extension. If B is projective, then A must be isomorphic to ΩC, and the fact that this extension is projective implies that the identity map on ΩC factors through a projective, meaning that $A \cong \Omega C$ is projective. But this is a contradiction. Hence, we may assume that B is not projective. Since A is indecomposable and nonprojective, ξ must be stable (i.e., $\underline{f} \neq 0$), for we have an exact sequence

$$(-, \underline{\Omega C}) \xrightarrow{0} (-, \underline{A}) \xrightarrow{(-,\underline{f})} (-, \underline{B})$$

in $\mathrm{mod}(\underline{\mathrm{gr}}_\Lambda)$. Thus,

$$0 \to (-, \underline{A}) \xrightarrow{(-,\underline{f})} (-, \underline{B}) \xrightarrow{(-,\underline{g})} (-, \underline{C}) \to F \to 0$$

is a minimal projective resolution of $F = \mathrm{coker}(-, \underline{g})$ in $\mathrm{mod}(\underline{\mathrm{gr}}_\Lambda)$. Applying $\bar{\alpha}$ gives a similar minimal projective resolution for $\bar{\alpha}F$ in $\mathrm{mod}(\underline{\mathrm{gr}}_\Gamma)$, from which we can obtain a minimal short exact sequence $0 \to K \xrightarrow{u} \alpha B \oplus P_0 \xrightarrow{v} \alpha C \to 0$ in gr_Γ with $\underline{v} = \alpha\underline{g}$. As in the proof of Theorem 3.4, there exists a split epimorphism $\pi : K \to \alpha A$ and taking the pushout of the above sequence via π yields a nonsplit stable extension $0 \to \alpha A \longrightarrow D \longrightarrow \alpha C \to 0$. Clearly, it suffices to show that this extension is projective. This can be seen by comparing the following two projective resolutions of $\bar{\alpha}F$ in $\mathrm{mod}(\underline{\mathrm{gr}}_\Gamma)$

$$\begin{tikzcd}
(-, \underline{\Omega\alpha C}) \arrow[r] \arrow[d] & (-, \underline{K}) \arrow[r] \arrow[d, "(-,\underline{\pi})"] & (-, \underline{\alpha B}) \arrow[r, "(-,\underline{v})"] \arrow[d, equal] & (-, \underline{\alpha C}) \arrow[r] \arrow[d, equal] & \bar{\alpha}F \arrow[r] \arrow[d, equal] & 0 . \\
0 \arrow[r] & (-, \underline{\alpha A}) \arrow[r] & (-, \underline{\alpha B}) \arrow[r, "(-,\alpha\underline{g})"] & (-, \underline{\alpha C}) \arrow[r] & \bar{\alpha}F \arrow[r] & 0
\end{tikzcd}$$

Looking at the leftmost square, we see that the composite $\Omega\alpha C \to K \xrightarrow{\pi} \alpha A$ factors through a projective, as required. □

Corollary 6.7. *With the above notation, $\mathrm{Ext}^1_\Lambda(C, A) = 0$ for f.g. nonprojective indecomposable Λ-modules A and C with C f.p. if and only if $\mathrm{Ext}^1_\Gamma(\alpha C, \alpha A) = 0$.*

Proof. Clearly, $\mathrm{Ext}^1_\Lambda(C, A) = 0$ if and only if $(\underline{\Omega C}, \underline{A}) = 0$ and $K(C, A) = 0$, and similarly over Γ. Since Λ and Γ are assumed to have no nodes, the vanishing Hom-group is equivalent to $(\underline{\Omega\alpha C}, \underline{\alpha A}) = 0$, while the second equality is equivalent to $K(\alpha C, \alpha A) = 0$ by the proposition. □

The following lemma and corollary will be important in the next section. Notice that they are trivial when Λ is right noetherian.

Lemma 6.8. *Let Λ be a locally artinian graded k-algebra, and suppose A is a f.g. Λ-module such that $\mathrm{Ext}^1_\Lambda(C, A) = 0$ for all f.p. Λ-modules C. Then A is injective.*

Proof. By Baer's criterion, for example, it suffices to show that $\mathrm{Ext}^1_\Lambda(C,A)=0$ for all f.g. Λ-modules C. So assume that C is f.g., and consider an extension of C by A, which we can complete to obtain the following commutative exact diagram.

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[r, "i"] \arrow[d, "f"] & P_C \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

To show that the extension on the bottom row splits, we must construct a map $g: P_c \to A$ such that $f = gi$. If $H \subseteq \Omega C$ is a f.g. submodule, $C' = P_C/H$ is f.p., and thus the restriction $f|_H$ extends to a map from P_C to A. Now, we can write ΩC as a direct limit $\varinjlim H_i$ of an ascending chain of f.g. submodules H_i, $i \in I$. Let f_i denote the restriction $f|_{H_i}$, and $g_i : P_C \to A$ the extension of f_i as above.

Notice that $g_j - g_i$ vanishes on H_i for any $j > i$. Thus, letting $K_i = \{\varphi : P_C \to A \mid \varphi|_{H_i} = 0\}$, we obtain a descending chain of k-submodules of $\mathrm{Hom}_\Lambda(P_C, A)$. Since P_C and A are locally finite graded modules, the set $\mathrm{Hom}_\Lambda(P_C, A)$ of degree 0 morphisms clearly has finite length over k, and hence satisfies DCC. Thus there exists some $n \in I$ such that $K_m = K_n$ for all $m \geq n$. In other words, if $\varphi|_{H_n} = 0$, then $\varphi|_{H_j} = 0$ for all $j \in I$, and hence $\varphi = 0$, as $\Omega C = \varinjlim H_i$. Therefore, $g_m|_{\Omega C} = g_n|_{\Omega C}$ for any $m > n$, and since each g_i is an extension of $f_i = f|_{H_i}$, g_n must coincide with f on all of ΩC. Hence g_n is our desired extension. □

Corollary 6.9. *Let α be a graded stable equivalence between algebras Λ and Γ without nodes, and assume either that Λ is right noetherian or that k is artinian. If A_Λ is f.g. and injective, then so is αA.*

Proof. If A is injective, we have $\mathrm{Ext}^1_\Lambda(C,A) = 0$ for all C f.p., and by the above corollary we have $\mathrm{Ext}^1_\Gamma(\alpha C, \alpha A) = 0$ for all f.p. Γ-modules αC. By the preceding lemma, αA is injective. □

We do not know whether or not the preceding corollary holds for *all* finitely generated locally finite k-algebras. In fact, this is currently the only obstruction to extending the proofs of Theorems 7.4 and 8.8 to this more general setting.

7. Almost split sequences and finite length modules

Our aim in this section is to show that a stable equivalence between locally finite graded algebras must take finite length modules to finite length modules. As the proof relies on the existence of almost split sequences beginning in modules of finite length, it is necessary to know that these are finitely copresented. We thus assume, throughout this section and the next, that all algebras considered are finitely generated as k-algebras. In particular, this ensures that the (right or left) simple modules, and thus all modules of finite length, are f.p. Since the duality D preserves finite length and takes f.p. modules to finitely copresented (f.cp.) modules, we see that all modules of finite length are also f.cp. Furthermore, notice that in this case a f.g. Λ-module has finite length if and only if it is finitely cogenerated (f.cg.), if and only if it is f.cp.

As mentioned earlier, given an indecomposable, noninjective Λ-module A of finite length, there is an almost split sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in Gr_Λ by [20]. Notice that, as DA has finite length, $C \cong \mathrm{Tr}DA$ is f.p. and hence B is f.p. as well. Therefore, such an almost split sequence in Gr_Λ, starting with a module of finite length, is in fact contained in gr_Λ (or even $\mathrm{f.p.gr}_\Lambda$). Provided that B is not projective, we can apply Theorem 4.1 to obtain a short exact sequence $0 \to \alpha A \xrightarrow{u} \alpha(\underline{B}) \oplus Q \xrightarrow{v} \alpha C \to 0$. We shall first show that this sequence is almost split in Gr_Γ, and once we know this it follows rather easily that αA has finite length, since it must be simultaneously f.g. and f.cp.

Theorem 7.1. *Let $0 \to A \xrightarrow{f} B \oplus P \xrightarrow{g} C \to 0$ be an almost split sequence in Gr_Λ, consisting of f.g. modules with P projective, and $\underline{f}$ nonzero. Then there is an almost split sequence $0 \to \alpha A \xrightarrow{u} \alpha B \oplus Q \xrightarrow{v} \alpha C \to 0$ in Gr_Γ with Q projective and $\underline{u} = \alpha(\underline{f}), \underline{v} = \alpha(\underline{g})$.*

Proof. The arguments in section X.1 of [6] work equally well in our setting. However, they show that the exact sequence of Γ-modules is almost split in gr_Γ. To see that it is in fact almost split in Gr_Γ, we apply the following lemma. □

Lemma 7.2. *Let $f: A \to B$ be an irreducible morphism in gr_Λ. Then f is irreducible in Gr_Λ.*

Proof. Suppose $f = ts$ for morphisms $s : A \to C$ and $t : C \to B$, and let $D \subseteq C$ be any f.g. submodule containing $s(A)$. By hypothesis, either $s : A \to D$ is a split monomorphism, or else $t|_D$ is a split epimorphism. The latter implies that t would also be a split epimorphism. Hence, we may assume that $s : A \to D$ is a split monomorphism for all f.g. $D \subseteq C$. Consequently, we may express s as a direct limit of split monomorphisms, and it follows that $s : A \to C$ is a pure monomorphism. But since A is f.g., we have $A \cong D(D(A))$, and hence A is pure injective (for example, use 1.2.2 of [20] to adapt 4.3 of [14] to the graded case). Thus s is a split monomorphism. $\square$

Remark. (8) With a little more work, it can be shown that this correspondence of almost split sequences is a special case of Theorem 4.1.

We have the following immediate corollary.

Corollary 7.3. *Let A be a noninjective indecomposable Λ-module of finite length. Then* $\mathrm{Tr}D(\alpha A) \cong \alpha(\mathrm{Tr}DA)$.

We can now prove the second fundamental result of this section.

Theorem 7.4. *Let $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ be an equivalence that commutes with the grading shift. If a nonprojective module A_Λ has finite length, then so does αA.*

Proof. Since separating nodes preserves modules of finite length, we may assume that neither algebra has nodes. Furthermore, we may assume that A is indecomposable. If A is injective, Corollary 6.9 implies that αA is also injective, and thus finitely cogenerated. Since αA is f.g., it must have finite length. Now assume that A is not injective. Thus we have an almost split sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, and $\underline{B} \neq 0$ by Proposition 2.3. By Theorem 7.1, we have an almost split sequence $0 \to \alpha A \xrightarrow{u} \alpha\underline{B} \oplus Q \xrightarrow{v} \alpha C \to 0$ in Gr_Γ. Thus αA is finitely copresented and finitely generated, and hence of finite length. $\square$

The following lemma will be of use in the next section, when we need to consider nodes, along with almost split sequences with projective middle terms.

Lemma 7.5. *Let $g : B \to C$ be a minimal right almost split morphism in gr_Λ with C f.p. Then the kernel of g is finitely cogenerated. In particular, if $\ker(g)$ is known to be f.g., then it must be of finite length.*

Proof. Consider the almost split sequence $0 \to D\mathrm{Tr}C \xrightarrow{s} E \xrightarrow{t} C \to 0$ in Gr_Λ. Since g does not split, it factors through t, say $g = tw$ for $w : B \to E$. Let $w' : A \to D\mathrm{Tr}C$ be the induced map, so that we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "w'"] & B \arrow[r, "g"] \arrow[d, "w"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D\mathrm{Tr}C \arrow[r, "s"] & E \arrow[r, "t"] & C \arrow[r] & 0.
\end{tikzcd}$$

We claim that w is injective. To see this, let $B' = \mathrm{im}\ w \subseteq E$, which is of course f.g. Thus g factors through $t|_{B'}$ and $t|_{B'}$ factors through g since g is right almost split in gr_Λ. Since g is right minimal it follows that w is an isomorphism onto B'. The snake lemma now implies that $w' : A \to D\mathrm{Tr}C$ is injective. Since $D\mathrm{Tr}C$ is f.c.p., A is f.c.g. $\square$

8. Algebras without socle

We now turn to the question of classifying algebras that are graded stably equivalent to algebras with zero socle. We keep the same assumptions on Λ as in the previous section. As a first step, we show that we do not need to worry about nodes in this case. Of course, it is a simple consequence of Proposition 2.3(v) that $\mathrm{soc}\,\Lambda = 0$ implies that Λ has no nodes or simple projectives.

Lemma 8.1. *Let $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ be an equivalence that commutes with the grading shift. If* $\mathrm{soc}\,\Lambda = 0$ *then* Γ *has no simple projectives.*

Proof. Suppose S_Γ is a simple projective. Then the almost split sequence $0 \to S \longrightarrow P \longrightarrow \mathrm{Tr}DS \to 0$ has projective middle term. Hence, $S \cong \Omega\mathrm{Tr}DS$ and Theorem 5.2 implies that $\Omega\alpha^{-1}\mathrm{Tr}DS$ is projective, as Λ has no nodes. Moreover, the arguments of the proof of Theorem 7.1 apply here to show that the projective cover $Q \to \alpha^{-1}\mathrm{Tr}DS$ is minimal right almost split in gr_Λ. Thus, by Lemma 7.5, its kernel $\Omega\alpha^{-1}\mathrm{Tr}DS$ has finite length. However, the existence of a finite length projective module contradicts soc $\Lambda = 0$. $\square$

Corollary 8.2. *Let $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ be an equivalence that commutes with the grading shift. If* soc $\Lambda = 0$ *then Γ has no nodes.*

Proof. If Γ has nodes, we may separate them to get an algebra Γ' with simple projectives that is graded stably equivalent to Λ. But this contradicts the previous lemma. $\square$

Proposition 8.3. *Let $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ be an equivalence that commutes with the grading shift. If* soc $\Lambda = 0$ *then* soc $\Gamma = 0$.

Proof. We know that neither Λ nor Γ has any nodes or simple projectives. Since soc $\Lambda = 0$, the syzygy of any f.g. nonprojective X_Λ does not have direct summands of finite length. Thus, for all f.g. nonprojective Y_Γ, $\underline{\Omega Y} \cong \underline{\alpha\Omega\alpha^{-1}Y}$ does not have finite length unless it is zero. Hence, if there exists a f.g. nonprojective Y_Γ with ΩY of finite length, then ΩY must be projective. Since Γ has no simple projectives, there exists a simple nonprojective submodule A of ΩY. Thus A has finite length and is the syzygy of the finite length nonprojective module $\Omega Y/A$. However, this implies that $\alpha^{-1}A$ is a finite length direct summand of the syzygy of $\alpha^{-1}(\Omega Y/A)$, a contradiction. $\square$

Corollary 8.4. *Let $\alpha : \underline{\mathrm{gr}}_\Lambda \to \underline{\mathrm{gr}}_\Gamma$ be an equivalence that commutes with the grading shift, and assume that* soc $\Lambda = 0$. *Then α induces an equivalence* $\mathrm{f.l.gr}_\Lambda \xrightarrow{\approx} \mathrm{f.l.gr}_\Gamma$.

Proof. Since soc $\Lambda = 0$, we have $\mathrm{Hom}_\Lambda(X, \Lambda[i]) = 0$ for all $i \in \mathbb{Z}$ and any X_Λ of finite length. Thus $\underline{\mathrm{Hom}}_\Lambda(X,Y) = \mathrm{Hom}_\Lambda(X,Y)$ for any finite length X and any f.g. Y, and similarly for Γ as soc $\Gamma = 0$ by the proposition. The vanishing socles of Λ and Γ also implies that neither algebra has a nonzero projective module of finite length. Therefore, for any finite length Λ-modules X and Y, we have

$$\mathrm{Hom}_\Lambda(X,Y) = \underline{\mathrm{Hom}}_\Lambda(X,Y) \xrightarrow{\cong} \underline{\mathrm{Hom}}_\Gamma(\alpha X, \alpha Y) = \mathrm{Hom}_\Gamma(\alpha X, \alpha Y),$$

from which we see that α gives an equivalence $\mathrm{f.l.gr}_\Lambda \approx \mathrm{f.l.gr}_\Gamma$. $\square$

We now investigate the consequences of an equivalence between the categories of finite length graded modules. If both algebras are basic, we will show that one algebra can be obtained from the other by regrading, and thus if both are concentrated in degree 0 they will simply be isomorphic. We will also see that we obtain an equivalence between the categories of all graded modules. We shall write $\mathbf{Hom}_\Lambda(X,Y)$ for the k-module of all homomorphisms from X to Y. Recall that if X is f.g. we have $\mathbf{Hom}_\Lambda(X,Y) \cong \oplus_{d\in\mathbb{Z}}\mathrm{Hom}_\Lambda(X[d],Y)$. We also write $\mathbf{End}_\Lambda(X)$ for the full endomorphism ring of X as an ungraded Λ-module. Again, as long as X is f.g., $\mathbf{End}_\Lambda(X) \cong \oplus_{d\in\mathbb{Z}}\mathrm{Hom}_\Lambda(X[d],X)$ is a graded ring that vanishes in sufficiently small degrees. Finally, recall that when we say an equivalence α commutes with the grading shifts, this means that there is an isomorphism of functors $\eta : \alpha \circ S_\Lambda \to S_\Gamma \circ \alpha$.

Lemma 8.5. *If $\alpha : \mathrm{f.l.gr}_\Lambda \to \mathrm{f.l.gr}_\Gamma$ is an equivalence that commutes with the grading shift, then α induces isomorphisms between full, graded endomorphism rings $\alpha : \mathbf{End}_\Lambda(X) \xrightarrow{\cong} \mathbf{End}_\Gamma(\alpha X)$ for any X of finite length.*

Proof. If $f : X[d] \to X$ is an endomorphism of degree d, $\alpha f : \alpha(X[d]) \to \alpha(X)$ and precomposing with the inverse of $\eta^d := \eta_X[d-1]\eta_{X[1]}[d-2]\cdots\eta_{X[d-1]} : \alpha(X[d]) \xrightarrow{\cong} (\alpha X)[d]$ yields an endomorphism of αX of degree d. Thus, for any $d \in \mathbb{Z}$, we can define

$$\varphi_d : \mathrm{Hom}_\Lambda(X[d], X) \to \mathrm{Hom}_\Gamma((\alpha X)[d], \alpha X)$$

by $\varphi_d(f) = \alpha(f) \circ (\eta^d)^{-1}$. Since η is an isomorphism of functors and α is an equivalence, each φ_d is an isomorphism. It thus remains only to see that the collection $\varphi = \{\varphi_d\}_{d\in\mathbb{Z}}$ defines a ring homomorphism. So let $f, g \in \mathbf{End}_\Lambda(X)$ be maps of degrees d, e respectively. The composite gf is a map of degree $d+e$, expressable as $g \circ (f[e]) : X[d+e] \to X[e] \to X$. We have the following commutative diagram

$$\begin{tikzcd}
\alpha(X[d+e]) \arrow[r, "\alpha(f[e])"] \arrow[d, "\eta^e"] & \alpha(X[e]) \arrow[r, "\alpha g"] \arrow[d, "\eta^e"] & \alpha X \\
(\alpha X[d])[e] \arrow[r, "(\alpha f)[e]"] \arrow[d, "\eta^d[e]"] & (\alpha X)[e] \arrow[ur, "\varphi_e(g)"'] & \\
(\alpha X)[d+e] \arrow[ur, "\varphi_d(f)[e]"'] & &
\end{tikzcd}$$

As $\eta^{d+e} = \eta^d[e] \circ \eta^e$, the large triangle defines $\varphi_{d+e}(g \circ f[e])$ along the hypotenuse. Hence, we have $\varphi_{d+e}(g \circ f[e]) = \varphi_e(g) \circ \varphi_d(f)[e]$, and it follows that φ is an isomorphism of graded rings. $\square$

We now apply Lemma 1.4 on completions of Λ-modules to show that the full graded endomorphism ring of a module M can be recovered as the inverse limit of the graded endomorphism rings of finite length quotients of M. Notice that we have natural graded ring homomorphisms $\pi_n : \mathbf{End}_\Lambda(M/MJ^n) \to \mathbf{End}_\Lambda(M/MJ^{n-1})$ and $\nu_n : \mathbf{End}_\Lambda(M) \to \mathbf{End}_\Lambda(M/MJ^n)$ for each $n \geq 1$. Clearly $\pi_n\nu_n = \nu_{n-1}$ for all $n \geq 1$, and thus we have an induced graded ring homomorphism $\varphi : \mathbf{End}_\Lambda(M) \to \varprojlim \mathbf{End}_\Lambda(M/MJ^n)$, where the inverse limit is taken in the category of graded k-algebras with degree-0 homomorphisms.

Lemma 8.6. *Let M be a f.g. graded Λ-module. Then the natural map $\varphi : \mathbf{End}_\Lambda(M) \to \varprojlim \mathbf{End}_\Lambda(M/MJ^n)$ is an isomorphism of graded k-algebras.*

Proof. For a fixed degree d, we have a sequence of isomorphisms by Lemma 1.4

$$\begin{aligned}
\mathrm{Hom}_\Lambda(M[d], M) &\cong \mathrm{Hom}_\Lambda(M[d], \varprojlim M/MJ^n) \\
&\cong \varprojlim \mathrm{Hom}_\Lambda(M[d], M/MJ^n) \\
&\cong \varprojlim \mathrm{Hom}_\Lambda((M/MJ^n)[d], M/MJ^n).
\end{aligned}$$

Moreover, this isomorphism coincides with the degree-d part of φ, as it is clear that

$$\left(\varprojlim \mathbf{End}_\Lambda(M/MJ^n)\right)_d \cong \varprojlim \mathbf{End}_\Lambda(M/MJ^n)_d \cong \varprojlim \mathrm{Hom}_\Lambda((M/MJ^n)[d], M/MJ^n). \quad \square$$

We shall say that two nonnegatively graded k-algebras Λ and Γ are *graded Morita equivalent* if there is an equivalence between Gr_Λ and Gr_Γ that commutes with the grading shift (this is called a "graded equivalence" in [10]). As in the nongraded setting, this is easily seen to be equivalent to Γ being isomorphic to the full graded endomorphism ring of some graded projective P_Λ that is a generator for $\mathrm{mod}(\Lambda)$ [10], and by Theorem 1.8.1 of [20] every locally finite graded k-algebra is graded Morita equivalent to a unique (up to isomorphism) basic locally finite graded k-algebra. Also, it is clear that a graded Morita equivalence between Λ and Γ induces equivalences between the corresponding subcategories of f.g. modules, finite length modules, etc. We now use a completion argument to show that an equivalence between the finite length module categories extends to an equivalence between the categories of all graded modules.

Proposition 8.7. *Suppose Λ and Γ are locally finite graded k-algebras such that* $\mathrm{f.l.gr}_\Lambda \approx \mathrm{f.l.gr}_\Gamma$ *by an equivalence commuting with the grading shift. Then Λ and Γ are graded Morita equivalent.*

Proof. Without loss of generality, we may assume that Λ and Γ are basic. Let α denote the equivalence of categories, and note that it is exact and takes simples to simples. Thus α preserves the Loewy lengths of modules. Let $S_1, \ldots, S_n$ denote the simple Λ-modules (up to isomorphism), concentrated in degree 0, and let $T_i = \alpha S_i$ for each i. While each T_i is a simple Γ-module, they are not necessarily all concentrated in the same degree. Furthermore, since α commutes with the grading shift, it is easy to see that any simple

Γ-module is isomorphic to a shift of exactly one T_i. Thus if $Q_1, \ldots, Q_n$ are the indecomposable graded projectives of Γ generated in degree 0, we have $T_i \cong (Q_i/Q_iJ_\Gamma)[d_i]$ for integers $d_i \in \mathbb{Z}$.

Now, for any $L \geq 1$, α induces an equivalence between the full, abelian subcategories of f.l.gr_Λ and f.l.gr_Γ consisting of modules of Loewy length at most L. Thus the indecomposable projective objects inside these categories correspond under α. Hence $\alpha(P_i/P_iJ_\Lambda^L)$ is isomorphic to some shift of $Q_i/Q_iJ_\Gamma^L$. In fact, we must have $\alpha(P_i/P_iJ_\Lambda^L) \cong (Q_i/Q_iJ_\Gamma^L)[d_i]$ since its top is concentrated in degree d_i, and the $Q_i/Q_iJ_\Gamma^L$ and their shifts are the indecomposable projectives in the category of Γ-modules of Loewy length at most L. Thus we get isomorphisms of endomorphism rings

$$\Lambda/J_\Lambda^L \cong \mathbf{End}_\Lambda(\oplus P_i/P_iJ_\Lambda^L) \cong \mathbf{End}_\Gamma(\oplus(Q_i/Q_iJ_\Gamma^L)[d_i]),$$

and the last endomorphism ring is isomorphic to Γ/J_Γ^L if we ignore the grading.

In this way we obtain a sequence of commutative diagrams of ring homomorphisms

$$\begin{tikzcd}
\Lambda/J_\Lambda^L \arrow[r, "\cong"] & \mathbf{End}_\Lambda(\oplus P_i/P_iJ_\Lambda^L) \arrow[r, "\cong", "\alpha"'] & \mathbf{End}_\Gamma(\oplus(Q_i/Q_iJ_\Gamma^L)[d_i]) \\
\Lambda/J_\Lambda^{L+1} \arrow[r, "\cong"] \arrow[u] & \mathbf{End}_\Lambda(\oplus P_i/P_iJ_\Lambda^{L+1}) \arrow[r, "\cong", "\alpha"'] \arrow[u] & \mathbf{End}_\Gamma(\oplus(Q_i/Q_iJ_\Gamma^{L+1})[d_i]) \arrow[u]
\end{tikzcd}$$

Taking inverse limits, we obtain isomorphisms of graded rings

$$\Lambda \cong \mathbf{End}_\Lambda(\oplus P_i) \cong \mathbf{End}_\Gamma(\oplus Q_i[d_i]),$$

and the latter is of course isomorphic to Γ if we forget the grading. In other words, we see that Λ is obtained as the graded endomorphism ring of the projective generator $\oplus_{i=1}^n Q_i[d_i]$ of Γ. Thus, the functor $\mathbf{Hom}_\Gamma(\oplus Q_i[d_i], -)$ induces an equivalence of categories between Gr_Γ and Gr_Λ [10]. $\square$

Remark. (9) As shown above, if Λ and Γ are basic and graded stably equivalent, then they must be isomorphic as ungraded k-algebras. We note, however, that the converse of this statement is rather far from being true. In fact, the gradings of Λ and Γ as in the proposition, can only differ in a very specific manner since both are assumed to be nonnegatively graded (cf. Corollary 5.11 in [10]).

Combining everything now gives our main result.

Theorem 8.8. *Assume either that Λ is right noetherian or that k is artinian, and let $\alpha : \underline{\text{gr}}_\Lambda \to \underline{\text{gr}}_\Gamma$ be an equivalence that commutes with the grading shift. If* $\text{soc}\ \Lambda = 0$*, then Λ and Γ are graded Morita equivalent.*

As $\text{soc}\ \Lambda$ may be characterized as the left annihilator of J_Λ, and $\mathfrak{m} = \text{rad}\ k$ is always contained in J_Λ, we obtain the following corollary by noting that $\mathfrak{m} \cdot \text{soc}\ \Lambda = 0$.

Corollary 8.9. *Suppose that k is a non-semisimple domain (i.e., not a field), and that Λ, viewed as a k-module, is torsionfree. Then* $\text{soc}\ \Lambda = 0$*, and consequently if Γ is graded stably equivalent to Λ, then Γ is graded Morita equivalent to Λ.*

As an even more elementary application, we can consider the case where Λ is a domain. This includes, in particular, all quantum polynomial rings (see [25] for a general definition).

For one final application we consider the preprojective algebras. We start with a connected bipartite graph Q that is not Dynkin, the edges of which we orient so that each vertex is either a sink or a source. For each arrow α of Q, we introduce a new arrow α^* with opposite orientation, and we call the quiver thus obtained $\hat{Q}$. The preprojective algebra of Q is then to defined to be the path algebra modulo relations $k\hat{Q}/I$ where I is the ideal generated by the sums, taken for each source i and each sink j of Q:

$$\sum_{s(\alpha)=i} \alpha\alpha^*, \quad \sum_{t(\alpha)=j} \alpha^*\alpha,$$

where the sums range over all arrows α of Q with source $s(\alpha) = i$, and with target $t(\alpha) = j$, respectively. In [19], it is shown that the preprojective algebras associated to non-Dynkin quivers arise as the Yoneda algebras of finite-dimensional self-injective Koszul algebras of Loewy length 3. As a result, it is easy to see that any such preprojective algebra Λ has $\text{soc}\ \Lambda = 0$. To see this, notice that if $\text{soc}\ \Lambda \neq 0$, there is a graded simple $S[i]$ that is a direct summand of J_Λ^n for some $n \geq 0$. By Koszul duality (cf. [11]), this corresponds to a nonzero projective direct summand of the n^{th} syzygy of $E(\Lambda)_0$, which cannot occur since $E(\Lambda)$ is self-injective.

Corollary 8.10. *Let Λ be the preprojective algebra associated to a non-Dynkin bipartite quiver. If Γ is graded stably equivalent to Λ, then Γ is graded Morita equivalent to Λ.*

Of course, the above corollary and its proof extend to any k-algebra that arises as the Koszul dual of a finite dimensional self-injective Koszul algebra. In [12] such algebras are called *Koszul generalized Auslander regular algebras*, and they also appear in [20, 25]. Notice that this class includes the algebras occuring as the Yoneda algebra $E(\Lambda)$ on the right hand side of the generalized BGG equivalence proved in [22]. It would be an interesting problem to determine how (if at all) the graded stable categories of these algebras are related to the derived categories mentioned there (or, equivalently, to the graded stable categories of their Koszul duals).

9. An example of graded stably equivalent nonartinian algebras

Earlier on we mentioned ways of obtaining graded stable equivalences between finite-dimensional graded algebras that were either concentrated in degree 0 or in degrees 0 and 1 (the latter arise as trivial extensions of derived equivalent algebras). While graded stable equivalences in this restricted setting are already quite interesting, we would still like to understand what types of examples exist between nonartinian rings (up to now, the only examples we have seen are given by construction and separation of nodes), and in what ways these might be connected to the finite-dimensional examples. For instance, if Λ and Γ are locally artinian graded algebras that are graded stably equivalent, are the artin algebras Λ_0 and Γ_0 stably equivalent? Conversely, given a stable equivalence between two artin algebras Λ_0 and Γ_0, can we construct a graded stable equivalence between certain graded extensions of these algebras? While these questions may not have affirmative answers in complete generality, they turn out to be the case in the example we now describe. This example is a simple adaptation of a method of Liu and Xi for constructing new stable equivalences of Morita type from old ones between finite-dimensional algebras [16]. It is based on a triangular matrix ring construction, and it turns out that the extending algebra does not need to be artinian. We start by stating a modification of the theorem of Liu and Xi without this assumption, and note that the proof is identical. To be consistent with their notation, we will work with left modules in this section.

Theorem 9.1 (cf. Theorem 4.2 in [16])**.** *Let A, B and C be algebras over a field k, with A and B finite-dimensional (with trivial grading) and C nonnegatively graded. Suppose that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. If R is a graded (A, C)-bimodule such that $M \otimes_B N \otimes_A R \cong R$ as graded (A, C)-bimodules and that the automorphism group of the module ${}_{B\otimes_k C^{\mathrm{op}}}(N \otimes_A R)$ is k^*, then there is a graded stable equivalence of Morita type between the triangular matrix algebras $\begin{pmatrix} A & R \\ 0 & C \end{pmatrix}$ and $\begin{pmatrix} B & N \otimes_A R \\ 0 & C \end{pmatrix}$.*

As an application, we can modify Example 3 in Section 6 of [16] by letting $C = k[x]$ with the usual grading (instead of $C = k$) and $R = {}_Ak_C$ concentrated in degree 0. Here, A and B are the principal blocks of kA_5 and kA_4, respectively, in characteristic 2. We thus obtain a graded stable equivalence between the triangular matrix rings $\Lambda = \begin{pmatrix} A & k \\ 0 & k[x] \end{pmatrix}$ and $\Gamma = \begin{pmatrix} B & k \\ 0 & k[x] \end{pmatrix}$, where k is the trivial module over A or B, and the unique graded simple module, concentrated in degree 0, over $k[x]$. In terms of quivers with relations, one simply adds a degree-1 loop x to the new vertex (denoted c and 3) of each quiver in Liu's and Xi's example, and the additional relation $\kappa x = 0$, where κ denotes the unique arrow leaving the new vertex in each case.

Similarly, if we modify this example by letting $C = F[[x]]$ for a field F and $R = {}_AF_C$, we obtain two stably equivalent noetherian algebras over the ring $k = F[[x]]$. In fact, the associated graded algebras of these algebras (with respect to the usual filtration on $F[[x]]$) are isomorphic to the algebras Λ and Γ of the previous example.

References

[1] M. Auslander. *Representation theory of Artin algebras I.* Comm. Algebra 1 (1974), no. 3, 177-268

[2] M. Auslander. *A survey of existence theorems for almost split sequences.* Representations of Algebras (Durham, 1985), 81-89, London Math. Soc. Lecture Note Ser. 116, Cambridge Univ. Press, Cambridge-New York, 1986.

[3] M. Auslander and I. Reiten. *Stable equivalence of Artin algebras.* Proc. of the Conf. on Orders, Group Rings and Related Topics (Ohio State Univ., Columbus, Ohio, 1972), 8-71, Lecture Notes in Math., Vol. 353, Springer, Berlin, 1973.

[4] M. Auslander and I. Reiten. *Representation theory of Artin algebras VI.* Comm. Algebra 6 (1978), no. 3, 267-300.

[5] M. Auslander and I. Reiten. *Stable equivalence of dualizing R-varieties.* Adv. Math. 12 (1974), no. 3, 306-366.

[6] M. Auslander, I. Reiten, S. Smalo. *Representation theory of Artin algebras.* Cambridge Studies in Advanced Mathematics 36, Cambridge Univ. Press, 1995.

[7] M. Auslander and O. Solberg. *Relative homology and representation theory I.* Comm. Algebra 21 (1993), no. 9, 2995-3031.

[8] I. N. Bernstein, I. M. Gel'fand, S. I. Gel'fand. *Algebraic bundles over $\mathbb{P}^n$ and problems of linear algebra.* Func. Anal. and Appl. 12 (1978), 212-214.

[9] B. Eckmann and P. J. Hilton. *Homotopy Groups of Maps and Exact Sequences.* Commentarii Mathematici Helvetici, vol. 34, fasc. 4, 1960, 271-304.

[10] R. Gordon and E. L. Green. *Graded Artin algebras.* J. Alg. 76, (1982), 111-137.

[11] E. L. Green and R. Martínez Villa. *Koszul and Yoneda algebras II.* Algebras and Modules II (Geiranger, 1996), 227-244, CMS Conf. Proc. 24, Amer. Math. Soc., Providence, RI, 1998.

[12] J. Y. Guo, R. Martínez Villa and M. Takane. *Koszul generalized Auslander regular algebras.* Algebras and Modules II (Geiranger, 1996), 227-244, CMS Conf. Proc. 24, Amer. Math. Soc., Providence, RI, 1998.

[13] D. Happel. *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras.* London Math. Soc. Lecture Note Ser. 119, Cambridge Univ. Press, Cambridge-New York, 1988.

[14] I. Herzog. *The Ziegler spectrum of a locally coherent Grothendieck category.* Proc. London Math. Soc. 74 (1997), no. 3, 503-558.

[15] T.Y. Lam. *A first course in noncommutative rings. Second edition.* Springer Graduate Texts in Mathematics 131, Springer, New York, 2001.

[16] Y. Liu and C. Xi. *Constructions of stable equivalences of Morita type for finite dimensional algebras I.* Trans. Amer. Math. Soc. 358 (2006), no. 6, 2537-2560.

[17] R. Martínez Villa. *Algebras stably equivalent to l-hereditary.* Representation theory II (Proc. ICRA II, Carleton Univ., Ottawa, Ont., 1979), 396-431, Lecture Notes in Math., 832, Springer, Berlin, 1980.

[18] R. Martínez Villa. *Properties that are left invariant under stable equivalence.* Comm. Algebra 18 (1990), no. 12, 4141-4169.

[19] R. Martínez Villa. *Applications of Koszul algebras: the preprojective algebra.* Representation theory of algebras (Cocoyoc, 1994), 487-504, CMS Conf. Proc. 18, Amer. Math. Soc., Providence, RI, 1996.

[20] R. Martínez Villa. *Graded, selfinjective and Koszul algebras.* J. Algebra 215 (1999), 34-72.

[21] R. Martínez Villa and G. Montaño Bermúdez. *Node deletion and stably equivalent Koszul algebras.* Comm. Algebra 33 (2005), no. 8, 2569-2585.

[22] R. Martínez Villa and M. Saorín. *Koszul equivalences and dualities.* Pacific Journal of Mathematics 214 (2004), no. 2, 359-378.

[23] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings. With the cooperation of L. W. Small. Revised edition.* Graduate Studies in Mathematics, 30. American Mathematical Society, Providence, RI, 2001.

[24] D. Orlov. *Derived categories of coherent sheaves and triangulated categories of singularities.* arXiv math.AG/0503632 (2005).

[25] S. P. Smith. *Some finite dimensional algebras related to elliptic curves.* Representation theory of algebras and related topics (Mexico City, 1994), 315-348, CMS Conf. Proc. 19, Amer. Math. Soc., Providence, RI, 1996.

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