Constructing minimal $\mathcal{P}^{<\infty}$-approximations over left serial algebras

Alex S. Dugas

Department of Mathematics, University of California, Berkeley, CA 94720, USA

Abstract

Let Λ be a finite dimensional left serial algebra over an algebraically closed field K. In this case, Burgess and Zimmermann Huisgen have shown that $\mathcal{P}^{<\infty}$, the full subcategory of Λ-mod consisting of the finitely generated Λ-modules of finite projective dimension, is contravariantly finite in Λ-mod. Moreover, they show that the minimal right $\mathcal{P}^{<\infty}$-approximations of the simple Λ-modules can be obtained by glueing together uniserials to form modules known as *saguaros*, and they state without proof an algorithm for constructing these approximations. We will review this algorithm and then demonstrate how a new notion of *graphical* morphisms between saguaros can be used to prove it.

Key words: left serial algebra, modules of finite projective dimension, contravariant finiteness, finitistic dimension.

1 Introduction

In this article we focus on finite dimensional left serial algebras over a field K, that is, algebras for which each indecomposable projective left module is uniserial. While representations of such algebras can be quite complex structurally, they provide interesting and relatively accessible examples of nontrivial homological phenomena. In [7], Zimmermann Huisgen shows that the left finitistic dimension of any left serial algebra Λ is finite (see also [6]). In fact, as any syzygy is shown to be isomorphic to a direct sum of uniserial modules (of which there are only finitely many up to isomorphism), the finitistic dimension is simply one greater than the largest finite projective dimension of a uniserial left Λ-module (except, of course, when fin.dim $\Lambda = 0$). While this proof yields a fairly straightforward way of calculating the finitistic dimension

Email address: `asdugas@math.berkeley.edu` (Alex S. Dugas).

in this case, it provides little information about the structure of Λ-modules of finite projective dimension. Further steps in this direction are explored in [4], where Burgess and Zimmermann Huisgen show that, over a left serial algebra Λ, the subcategory $\mathcal{P}^{<\infty}$ of all finitely generated Λ-modules of finite projective dimension is contravariantly finite in Λ-mod. In this case, a well-known result of Auslander and Reiten states that the minimal right $\mathcal{P}^{<\infty}$-approximations of the simple Λ-modules form a finite set of "building blocks" for all modules in $\mathcal{P}^{<\infty}$, and it follows easily that fin.dim Λ can be computed as the supremum of the projective dimensions of these approximations [2]. It thus becomes quite desirable to be able to construct the minimal $\mathcal{P}^{<\infty}$-approximations of the simple modules. In [4], it is shown that they can be obtained by "glueing" together uniserial modules along isomorphic submodules, and this structural description is achieved through a more general study of modules obtained in this fashion, which are named *saguaros* after the cactus-shaped diagrams associated to them.

The goal of this article is to present and prove an algorithm (stated originally in [4]) for the construction of those saguaros that arise as the minimal right $\mathcal{P}^{<\infty}$-approximations of the simple Λ-modules when Λ is left serial. Not only does this algorithm yield another method of computing the finitistic dimensions of left serial algebras, but it also presents a key step towards a completely explicit structural description of the modules of finite projective dimension over such algebras. At the same time, in our proof we encounter a new tool for studying morphisms between modules which have diagrams in the sense of [1] or [5].

2 Review of Saguaros

Throughout this article Λ shall denote a basic, finite dimensional left serial algebra over an algebraically closed field K. Furthermore, we fix a presentation of Λ as the path algebra of a finite quiver Γ modulo an admissible ideal I of relations. The condition that Λ is left serial is equivalent to the condition that no vertex of Γ is the source of more than one arrow. It follows easily from this property of Γ that the ideal I is generated by paths, and hence Λ is a monomial relation algebra.

Our notation shall closely follow that of [4], and we refer the reader to [3] for general facts and terminology from representation theory. As in [4], we fix a *normed K-basis* for Λ as follows. Let $\mathbf{B}_i$ consist of the paths of length i in Γ that are not contained in I. We shall always identify such paths with their images in $\Lambda = K\Gamma/I$. Note that, in particular, $\mathbf{B}_0$ is a set of orthogonal primitive idempotents in Λ with sum 1, corresponding to the vertices of Γ. The set $\mathbf{B} = \cup_{i\geq 0}\mathbf{B}_i$ is a K-basis for Λ, while $\mathbf{B}^* = \cup_{i\geq 1}\mathbf{B}_i$ is a K-basis for $J = \text{rad}\ \Lambda$. Moreover, these bases are *normed* in the sense that any element

$\alpha \in \mathbf{B}$ is equal to $e\alpha$ for a unique primitive idempotent $e \in \mathbf{B}_0$. In this case, we also say that α is *normed* by e. We may now give the precise definition of saguaros as in [4].

Definition 2.1 *Let $T_1, \ldots, T_m$ be nonzero uniserial Λ-modules. A Λ-module T is a* ***saguaro*** *on $(T_1, \ldots, T_m)$, relative to the K-basis $\mathbf{B}$, if*

(i) $T \cong (\oplus_{i=1}^{m} T_i)/\sum_{i=1}^{m-1} \Lambda(b_i t_i - c_i t_{i+1})$, *where for each i, $t_i \in T_i$ is a generator normed by some primitive idempotent in $\mathbf{B}_0$, and $b_i, c_i \in \mathbf{B}^*$ are such that $b_i t_i \neq 0$ and $c_i t_{i+1} \neq 0$; and*

(ii) each T_j embeds canonically in T via the composite of natural maps

$$T_j \to \oplus_{i=1}^{m} T_i \to T.$$

Suppose that T is a saguaro on $(T_1, \ldots, T_m)$. The uniserial modules T_i will be called the *trunks* of T, and the canonical image of T_i in T will be denoted $\hat{T}_i$. If $T_i = \Lambda t_i$ for t_i as in the definition, we shall write $\hat{t}_i$ for the image of t_i in T, and we refer to the sequence $\hat{t}_1, \ldots, \hat{t}_m$ as a *canonical sequence of top elements* for T. (Notice that there is nothing present in these definitions which requires Λ to be left serial. The reader may find a completely general development of saguaros in [4]).

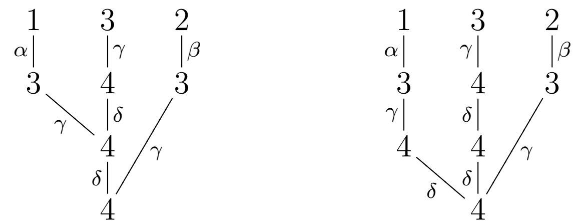
Saguaros can be conveniently visualized with the help of labeled and layered graphs akin to those studied by Alperin [1] and Fuller [5]. We give one example here and refer the reader to [4] for precise definitions and more examples. We point out that such a graph for the saguaro T does in general depend on the choice of a canonical sequence of top elements. For our example, let Λ be the left serial algebra with quiver

$$\begin{tikzcd} & 2 \arrow[d, "\beta"] & \\ 1 \arrow[r, "\alpha"'] & 3 \arrow[r, "\gamma"'] & 4 \arrow[loop right, "\delta"] \end{tikzcd}$$

and indecomposable projectives having graphs

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ |\,\alpha & |\,\beta & |\,\gamma & |\,\delta \\ 3 & 3 & 4 & 4 \\ |\,\gamma & |\,\gamma & |\,\delta & |\,\delta \\ 4 & 4 & 4 & 4. \\ |\,\delta & & |\,\delta & \\ 4 & & 4 & \end{array}$$

The following graphs represent two distinct saguaros with the projective modules $\Lambda e_1, \Lambda e_3$ and Λe_2 as trunks.



The existence of these graphical presentations of saguaros makes it quite easy to visualize their structures. In particular, the numbers (nodes) in these graphs represent the isomorphism types of the simple composition factors of the associated modules, while the rows illustrate the Loewy layers of these modules. However, given graphs of two saguaros X and Y, the morphisms from X to Y cannot be visualized so easily. This observation motivates us to introduce the following notions for morphisms between saguaros.

Definition 2.2 *Let X and Y be saguaros with fixed sequences of top elements $\{\hat{x}_i\}_{i=1}^n$ and $\{\hat{y}_j\}_{j=1}^m$ respectively. We call a morphism $f : X \to Y$* ***graphical*** *(with respect to the given sequences of top elements) if for each i there exists an index j_i and a path $a_i \in \mathbf{B}$ such that $f(\hat{x}_i) = a_i \hat{y}_{j_i}$. If, in addition, f is a monomorphism, we shall refer to it as a* ***graphical embedding****.*

Of course, this definition makes sense for any morphism between modules admitting graphs in the above sense. In fact, graphical morphisms may be viewed as those morphisms induced, in the natural way, by the homomorphisms of diagrams considered by Fuller [5]. Notice, however, that the definition—since it deals with modules rather than diagrams—depends heavily upon the selected sequences of top elements for the two saguaros. A morphism f may very well be graphical with respect to one choice of a canonical sequence of top elements, but not with respect to another. Furthermore, as the following example illustrates, there exist maps between saguaros that are not graphical with respect to any sequences of top elements. If Λ is the algebra introduced in the above example, let $X = \Lambda e_3$ and let Y be the saguaro illustrated above on the left. We can define $f : X \to Y$ by sending the top element $\hat{x}_1 \in X$ to $\alpha\hat{y}_1 + \hat{y}_2 \in Y$ where $\hat{y}_1, \hat{y}_2, \hat{y}_3$ are the top elements of Y yielding the above graph. In order to make f into a graphical morphism, we would need to choose $\hat{y}_1, \alpha\hat{y}_1 + \hat{y}_2, \hat{y}_3$ as our sequence of top elements for Y. However, one easily checks that Y is not a saguaro with respect to this sequence of top elements.

Nevertheless, graphical maps are convenient for they respect the structure depicted in the graphs of saguaros. In particular, if $f : X \to Y$ is a graphical embedding, it identifies the graph of X with a subgraph of the graph of Y (where everything is with respect to the same fixed canonical sequences of top elements $\{\hat{x}_i\}_{i=1}^n$ for X and $\{\hat{y}_j\}_{j=1}^m$ for Y). Furthermore, if we identify X with its image in Y, then the quotient Y/X is a finite direct sum of saguaros with canonical sequences of top elements given by the nonzero residue classes of the $\hat{y}_j$, and the graphs for these saguaros are obtained by removing the (sub)graph of X from the graph of Y. Another important fact we shall use is that a composite of graphical maps is again graphical.

We call a saguaro T *reduced* if it has simple socle. By Observation 3.8 in [4], over left serial algebras this condition admits the following simple graphical characterization: T is reduced if and only if, in its graph, no two edges entering a given node from above have the same label. For example, a quick glance at the graphs of saguaros pictured above now reveals that the one on the left is reduced, whereas the one on the right is not. Also of key importance is the fact that over a left serial algebra there are only finitely many isomorphism classes of reduced saguaros ([4], Theorem 5.1). Finally, for the reader's convenience, we conclude this section with a summary of the main result of [4], establishing the contravariant finiteness of $\mathcal{P}^{<\infty}$ in Λ-mod.

Theorem 2.3 (cf. [4], Theorems 5.2 and 5.3) *If Λ is a finite dimensional left serial algebra over an algebraically closed field K, then minimal right $\mathcal{P}^{<\infty}$-approximations of the simple left Λ-modules exist, and they are reduced saguaros. Moreover, if $S = \Lambda e/Je$ is a simple Λ-module with $e \in \mathbf{B}_0$, and $C \subsetneq Je$ is chosen to have maximal length such that $T_* := \Lambda e/C \in \mathcal{P}^{<\infty}$, then let T be any saguaro of maximal K-dimension among all reduced saguaros in $\mathcal{P}^{<\infty}$having T_* as a trunk. If $T_* = T_j$ is the j^{th} trunk of T, a minimal $\mathcal{P}^{<\infty}$-approximation of S is given by the morphism $f : T \to S$ that sends $\hat{t}_j$ to $e + Je \in S$ and $\hat{t}_i$ to 0 for all $i \neq j$. In particular, T is unique up to isomorphism.*

3 The Algorithm

We begin by giving an informal visual summary of Algorithm 7.1 of [4], and then we proceed to reformulate the algorithm more rigorously. We fix a simple Λ-module $S = \Lambda e/Je$, where $e \in \mathbf{B}_0$ is a primitive idempotent and $J = \mathrm{rad}\,\Lambda$. Let $g : U \to S$ be the minimal $\mathcal{P}^{<\infty}$-approximation of S as described in Theorem 2.3 above. We shall use T to denote the final output of the algorithm described below. Our goal will then be to show that $T \cong U$.

The first step of the algorithm is to find $C \subsetneq Je$ of maximal length such that $T_* = \Lambda e/C \in \mathcal{P}^{<\infty}$, for we know by Theorem 2.3 that the uniserial module T_* must be a trunk of U. Then $T^{(1)} = T_*$ represents our first approximation of U. The next step is to attach another trunk T_2 to the highest possible node of JT_* such that the resulting saguaro is reduced and of finite projective dimension. If there are multiple trunks that can be attached in this way, we choose one yielding a "branch" of maximal length in the new saguaro $T^{(2)}$. If, on the other hand, no trunk T_2 can be attached to give a reduced saguaro of finite projective dimension, the algorithm terminates and sets $T = T^{(1)}$.

The next step is to attach a trunk T_3 to the highest possible node of $J\hat{T}_2 \subseteq T^{(2)}$ such that the resulting saguaro is reduced and of finite projective dimension. As before, if there are multiple options for T_3 we choose one yielding a branch of maximal length. The algorithm continues in this manner—at each stage we attach a new trunk to the highest possible node of the last trunk in the same way as described above—until we get a saguaro $T^{(n)}$ to which no trunk can be attached in this way to obtain a reduced saguaro of finite projective dimension. At this stage, the algorithm terminates and sets $T = T^{(n)}$.

Before stating the algorithm more precisely, we review the process of attaching trunks to a saguaro. For a saguaro T on trunks $(T_1, \dots, T_n)$, we introduce a set $\mathcal{A}(T)$ whose elements will correspond to the different ways of attaching a trunk to T along the last trunk T_n. Without loss of generality, we may assume $T = (\oplus_{i=1}^n T_i)/\sum_{i=1}^{n-1} \Lambda(b_i t_i - c_i t_{i+1})$ with notation as in the definition of saguaros. Now define

$$\mathcal{A}(T) = \{(b, c) \in (\mathbf{B}^*)^2 \mid bt_n \neq 0,\ \operatorname{ann}_\ell(c) \subseteq \operatorname{ann}_\ell(bt_n)\}.$$

Lemma 3.1 *Let T be a saguaro on $(T_1, \dots, T_n)$ and suppose $(b, c) \in \mathcal{A}(T)$. Then*

(a) The elements b and c of $\mathbf{B}^*$ are normed on the left by the same primitive idempotent $e' \in \mathbf{B}_0$.
(b) If $T' = \Lambda e''/\operatorname{ann}_\ell(bt_n)c = \Lambda t'$, where $ce'' = c$ for $e'' \in \mathbf{B}_0$, then $\operatorname{ann}_\ell(ct') = \operatorname{ann}_\ell(bt_n)$.

Thus $T(b, c) := (T \oplus T')/\Lambda(b\hat{t}_n - ct')$ is a saguaro with trunks $(T_1, \dots, T_n, T')$.

Proof. For (a), suppose $e'c = c$ and $e''b = b$ for primitive idempotents $e', e'' \in \mathbf{B}_0$. Then $1 - e' \in \operatorname{ann}_\ell(c) \subseteq \operatorname{ann}_\ell(bt_n)$, and $e'bt_n = bt_n \neq 0$. Since $e'b$ would be zero if e' was different from e'', we must have $e' = e''$. For (b), notice that $\operatorname{ann}_\ell(ct') = \{\lambda \in \Lambda \mid \lambda c \in \operatorname{ann}_\ell(bt_n)c\}$, which certainly contains $\operatorname{ann}_\ell(bt_n)$. Meanwhile, if $\lambda c = rc$ for some $r \in \operatorname{ann}_\ell(bt_n)$, then $\lambda - r \in \operatorname{ann}_\ell(c) \subseteq \operatorname{ann}_\ell(bt_n)$, and hence $\lambda \in \operatorname{ann}_\ell(bt_n)$.

To check that $T(b, c)$ is a saguaro on the given trunks it suffices to observe that the canonical maps from T_n and T' to $T(b, c)$ are injective. But this follows from part (b). □

Remark. The saguaro $T(b, c)$ may also be defined by the short exact sequence

$$0 \to \Lambda c \xrightarrow{\phi} T \oplus \Lambda e'' \longrightarrow T(b, c) \to 0,$$

where $\phi(c) = (b\hat{t_n}, c)$. In particular, this shows that if pdim $T < \infty$, then pdim $T(b, c)$ is finite if and only if pdim Λc is finite.

Algorithm 3.2

Step 1. Find $T_* = \Lambda e/C$ where $C \subseteq Je$ has maximal length such that $T_* \in \mathcal{P}^{<\infty}$. That is, T_* is chosen to be the smallest nonzero quotient of Λe which has finite projective dimension. Set $T_1 = T_* = \Lambda t_1$ where t_1 is the residue class of e in T_1.

Inductive Step. Let $T^{(n)}$ denote the intermediate saguaro obtained after n steps. Then $T^{(n)}$ will have n trunks $T_1, \dots, T_n$, and it will be of the form

$$T^{(n)} = \bigoplus_{i=1}^{n} T_i \ / \ \sum_{i=1}^{n-1} \Lambda(b_i t_i - c_i t_{i+1}).$$

We want to attach a trunk to the highest possible node of $J\hat{T}_n \subset T^{(n)}$ such that the resulting saguaro is still reduced and of finite projective dimension. In order to describe how this will be done, we define the set

$$\mathcal{A}_n^* = \{(b, c) \in \mathcal{A}(T^{(n)}) \mid \text{pdim } \Lambda c < \infty, \text{ and } T(b, c) \text{ is reduced}\}.$$

If $\mathcal{A}_n^*$ is empty, then the algorithm terminates and sets $T = T^{(n)}$. Otherwise, we pick a pair $(b_n, c_n) \in \mathcal{A}_n^*$ such that the length of b_n is minimal, and such that the length of c_n is then maximal for this choice of b_n. We now define $T^{(n+1)} = T(b_n, c_n)$, renaming the new trunk T' as T_{n+1} and its generator t' as t_{n+1}.

It is clear by construction that all the intermediate saguaros encountered in this algorithm are reduced. As a result, the fact that there are only finitely many nonisomorphic reduced saguaros implies that the algorithm always terminates. Moreover, the condition $\text{pdim}(\Lambda c) < \infty$, together with the choice of T_1, ensures that all the intermediate saguaros have finite projective dimension by the remark that precedes the algorithm. We also point out that the algorithm must terminate with $T^{(n)}$ in case $T^{(n)} \cong U$, since it follows from Theorem 2.3 that the length of U is at least as large as the length of any intermediate saguaro. Thus the above algorithm always yields a reduced saguaro T of finite projective dimension, and it remains only to verify that this T is isomorphic to the minimal right $\mathcal{P}^{<\infty}$-approximation U of S. Our strategy will be to show that if the intermediate saguaro $T^{(n)}$ is not isomorphic to U, then the algorithm does not terminate at this stage.

4 Constructing Graphical Maps

Continuing the notation of the last section, the saguaro $T = T^{(n)}$, with which the algorithm terminates, maps to S via the trunk T_*. If we label this map f, we have $f(\hat{t}_1) = e + Je \in S$ and $f(\hat{t}_i) = 0$ for $i = 2, \dots, n$. Since T has finite projective dimension, f must factor through the minimal right $\mathcal{P}^{<\infty}$-approximation $g : U \to S$. The principal step in our proof of the algorithm is to show that the induced map from T to U can be replaced with a graphical embedding. We will then show that $\mathcal{A}_n^* = \emptyset$ implies that this graphical embedding must be an isomorphism.

We begin by stating a slight generalization of a result proven in the verification of Claim 2 of the proof of Theorem 5.3 of [4]. First, notice that, by construction, the trunk T_* has the property that all of its proper, nonzero quotients (and hence also all of its proper, nonzero submodules) have infinite projective dimensions, whereas T_* itself has finite projective dimension. We shall henceforth call any module with this property $\mathcal{P}^{<\infty}$-*minimal*. Using this terminology we have the following.

Proposition 4.1 (cf. [4]) *Suppose X is a saguaro of finite projective dimension which contains a $\mathcal{P}^{<\infty}$-minimal trunk X_i. If X is not reduced then X has a simple submodule with finite projective dimension. Moreover, this submodule must have the form $\Lambda(a_j\hat{x}_j - a_k\hat{x}_k) \cong \Lambda e'/Je'$ where $\hat{X}_j \cap \hat{X}_k = \Lambda\alpha a_j\hat{x}_j = \Lambda\alpha a_k\hat{x}_k$ for some $a_j, a_k \in \mathbf{B}$ and some arrow α, and where $e' \in \mathbf{B}_0$ is the primitive idempotent corresponding to the tail of α.*

We now begin our construction of a graphical embedding in a slightly more general context than necessary, but the reader is invited to replace X and Y with T and U respectively. We recall that the *amalgam* $X \vee Y[X_*]$ of two saguaros X and Y along a common trunk X_* is defined as the pushout of the canonical inclusions $X_* \to X$ and $X_* \to Y$. Equivalently, it can be defined by the short exact sequence

$$0 \to X_* \longrightarrow X \oplus Y \longrightarrow X \vee Y[X_*] \to 0,$$

where the first map is induced by the canonical inclusions mentioned above.

Proposition 4.2 *Let X and Y be two (nonsimple) reduced saguaros of finite projective dimension which have a common $\mathcal{P}^{<\infty}$-minimal trunk X_*. Let $V = X \vee Y[X_*]$ be their amalgam along this trunk. Then V has a quotient W which is a reduced saguaro of finite projective dimension with trunk X_*, and in which both X and Y embed graphically via the composites of the inclusions into V with the projection of V onto W.*

Proof. Let W be any quotient of V of minimal length among all quotients V'

of V satisfying the following properties:

(1) V' is a saguaro of finite projective dimension;
(2) the natural map $V \to V'$ is graphical;
(3) V' has a trunk isomorphic to X_*;
(4) the composites $X \to V \to V'$ and $Y \to V \to V'$ are injective.

Notice that we can always find such a W, since V itself satisfies these four properties and V has finite length. Furthermore, it follows immediately from (2) and (4), along with the fact that composites of graphical maps are graphical, that the composites in (4) are graphical embeddings. Thus, all it remains to show is that W is reduced.

We suppose, to the contrary, that W is not reduced. Then, by Proposition 4.1, W has a simple submodule N of finite projective dimension. Moreover $N = \Lambda(a_j \hat{w}_j - a_k \hat{w}_k) \cong \Lambda e'/Je'$ where $\hat{W}_j \cap \hat{W}_k = \Lambda \alpha a_j \hat{w}_j = \Lambda \alpha a_k \hat{w}_k$ with notation as in Proposition 4.1. After a reordering of the trunks of W (by Observation 3.6 of [4]) so that $k = j+1$, factoring out this simple submodule gives a new saguaro W' whose graph can be obtained from the graph of W by simply identifying the two edges labeled α (see the figure on page 87 of [4]). We now verify that W' also satisfies the four properties listed above, thereby contradicting the minimality of W.

(1) We have already seen that W' is a saguaro. Since both W and N have finite projective dimensions, so does $W' \cong W/N$.
(2) The images in W' of the top elements $\hat{w}_i$ of W are still top elements, with the possible exception of $\hat{w}_j$ or $\hat{w}_{j+1}$. Moreover, one of the images of $\hat{w}_j$ and $\hat{w}_{j+1}$ can only fail to be a top element for W' if exactly one of a_j, a_{j+1} is in $\mathbf{B}_0$. However, in this case the images of $\hat{w}_j$ and $\hat{w}_{j+1}$ will be a top element $\hat{w}'$ of V' and an element of the form $a\hat{w}'$ for a path $a \in \mathbf{B}^*$. In particular, the natural map $W \to W'$ is graphical, and thus so is the composite $V \to W \to W'$.
(3) Since X_* is $\mathcal{P}^{<\infty}$-minimal, soc X_* cannot have finite projective dimension. Thus soc X_*, and hence X_*, intersects N trivially in W, and it follows that the image of X_* in W' is a submodule isomorphic to X_*. The only way this image could fail to be a trunk of W' is if the top element $\hat{w}_i \in W$ generating the trunk $\hat{W}_i \cong X_*$ happens to be $\hat{w}_j$ or $\hat{w}_{j+1}$ and $a_i = e' \in \mathbf{B}_0$, by the above remarks. However, if this were the case, we would have $N \cong \Lambda e'/Je' \cong X_*/JX_*$, which contradicts the $\mathcal{P}^{<\infty}$-minimality of X_*.
(4) The composite $X \to V \to W \to W'$ is injective since (by the argument in (3)) its restriction to $\hat{X}_* \subseteq X$ is injective, and the trunk $\hat{X}_*$ contains soc X. The same statement for Y is proved similarly.

Thus, as the existence of W' contradicts the minimality of W, W is in fact reduced as required. □

Now consider the minimal right $\mathcal{P}^{<\infty}$-approximation $U \xrightarrow{g} S$. We know that U is a reduced saguaro with a trunk U_i isomorphic to the uniserial module T_* appearing in the algorithm. Furthermore, the map g is determined by $g(\hat{u}_i) = e + Je \in \Lambda e/Je$ and $g(\hat{u}_j) = 0$ for all $j \neq i$. Thus, if we apply the above proposition with U in place of Y and T_* in place of X_*, we obtain a reduced saguaro $W \in \mathcal{P}^{<\infty}$ and a graphical embedding $h : U \hookrightarrow W$ with the property that $h(\hat{u}_i)$ is a top element of W generating a trunk isomorphic to T_*. However, this implies that g factors through a map $W \to S$ which sends $h(\hat{u}_i)$ to $e + Je$ and all other members of the canonical sequence of top elements of W to zero. By the uniqueness of minimal right $\mathcal{P}^{<\infty}$-approximations, $h : U \to W$ must be an isomorphism. Thus we obtain the following corollary.

Corollary 4.3 *Let X be a reduced saguaro of finite projective dimension with a trunk $X_i \cong T_*$, and let U be as above. Then X admits a graphical embedding into U (with respect to any fixed choice of canonical sequences of top elements for X and U).*

We can now prove the validity of Algorithm 3.2.

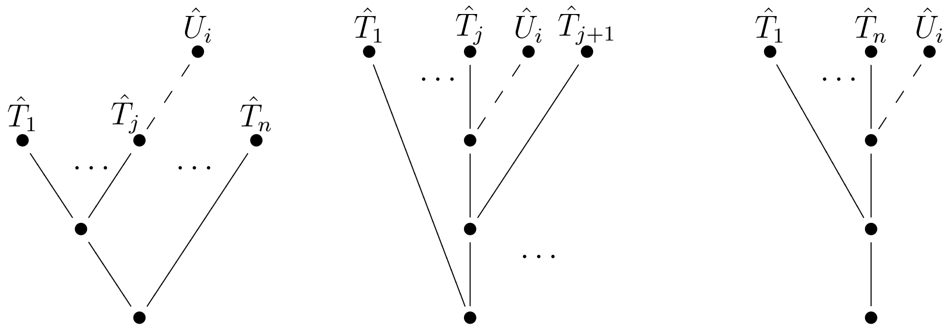
Theorem 4.4 *If the saguaro $T^{(n)}$ obtained in the n^{th} stage of Algorithm 3.2 is not isomorphic to the minimal right $\mathcal{P}^{<\infty}$-approximation U of the simple module S, then the set $\mathcal{A}_n^*$ is nonempty and the algorithm does not terminate. In particular, the final output T of the algorithm is isomorphic to U.*

Proof. Clearly, it suffices to prove the first statement. Thus, suppose that $T = T^{(n)}$ is not isomorphic to U. Since T is a reduced saguaro in $\mathcal{P}^{<\infty}$ with a trunk $T_1 \cong T_*$, the preceding corollary gives a graphical embedding $h : T \hookrightarrow U$. Since $U, T \in \mathcal{P}^{<\infty}$, and h is a graphical embedding but not an isomorphism, coker h is a nonzero direct sum of saguaros of finite projective dimensions. Moreover, coker h has a canonical sequence of top elements consisting of the nonzero residue classes $\hat{u}_i + T$ (we henceforth identify T with its image in U). By Observation 3.10 of [4], there exists some $\hat{u}_i \notin T$ such that $\Lambda(\hat{u}_i + T) \subseteq U/T$ has finite projective dimension. Now

$$\frac{\Lambda\hat{u}_i + T}{T} \cong \frac{\Lambda\hat{u}_i}{T \cap \Lambda\hat{u}_i} \cong \Lambda(\hat{u}_i + T) \in \mathcal{P}^{<\infty}.$$

Since T also belongs to $\mathcal{P}^{<\infty}$, it follows that $\Lambda\hat{u}_i + T \in \mathcal{P}^{<\infty}$. But clearly the latter is a saguaro, as it is just the submodule of U generated by T and $\Lambda\hat{u}_i$, and it is reduced since U is.

It simply remains to show that this saguaro $U' := \Lambda\hat{u}_i + T$ can be obtained from T by attaching a trunk as in the algorithm, i.e., that it is isomorphic to some $T(b, c)$ with $(b, c) \in \mathcal{A}_n^*$. Clearly it can be obtained from T by attaching the trunk $\Lambda \hat{u}_i$, but we must verify that $\Lambda \hat{u}_i$ can be attached to T along the trunk $\hat{T}_n$. Graphically, we have three possibilities to consider, which we depict below. The following graphs do not show every trunk and node of the corresponding saguaros; rather, their purpose is to illustrate the different ways in which the trunk $\hat{U}_i := \Lambda \hat{u}_i$ might intersect T.



$\hat{U}_i \cap T = \hat{T}_j$ $\qquad$ $\hat{U}_i \cap T = \hat{U}_i \cap \hat{T}_j \subset \hat{T}_j$ $\qquad$ $\hat{U}_i \cap T = \hat{U}_i \cap \hat{T}_n \subset \hat{T}_n$

$(j \leq n)$ $\qquad$ $(j < n)$

However, according to the algorithm, the first two diagrams cannot occur. The first would contradict the choice of a longest branch in the j^{th} stage of the algorithm, while the second would contradict the choice of the highest node at which to attach the $(j+1)^{th}$ trunk. In either case, the saguaro $T^{(j)} + \Lambda \hat{u}_i$ has finite projective dimension since it is an extension of $\Lambda(\hat{u}_i + T)$ by $T^{(j)}$, and it is clearly reduced since it is a submodule of U. To be more precise, the index j encountered above is determined, without any reference to graphs, as the largest index for which $T \cap \Lambda \hat{u}_i = \hat{T}_j \cap \Lambda \hat{u}_i$. Moreover, we can now check algebraically that $T^{(j)} + \Lambda \hat{u}_i$ has finite projective dimension, for

$$\frac{T^{(j)} + \Lambda \hat{u}_i}{T^{(j)}} \cong \frac{\Lambda \hat{u}_i}{T^{(j)} \cap \Lambda \hat{u}_i} \cong \frac{\Lambda \hat{u}_i}{\hat{T}_j \cap \Lambda \hat{u}_i} \cong \frac{\Lambda \hat{u}_i}{T \cap \Lambda \hat{u}_i} \in \mathcal{P}^{<\infty}.$$

Therefore, U' must resemble the third diagram above, that is, $\Lambda \hat{u}_i \cap T = \Lambda \hat{u}_i \cap \hat{T}_n$. Hence, we may choose $b, c \in \mathbf{B}$ such that $T \cap \Lambda \hat{u}_i = \hat{T}_n \cap \Lambda \hat{u}_i = \Lambda c \hat{u}_i = \Lambda b \hat{t}_n$. Clearly, our choice of b and c satisfies $\mathrm{ann}_\ell(c) \subseteq \mathrm{ann}_\ell(c\hat{u}_i) = \mathrm{ann}_\ell(b\hat{t}_n)$. The maximal choice of c_{n-1} in the algorithmic construction of T assures us that b has positive length as an element of $\mathbf{B}$, while the fact that $\hat{u}_i \notin T$ guarantees c has positive length as well. Hence, $T + \Lambda \hat{u}_i \cong T(b, c)$ and since this module is a reduced saguaro in $\mathcal{P}^{<\infty}$, $\mathcal{A}_n^*$ is nonempty, as required, and the algorithm does not halt at this stage. $\square$

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