

# W Week 2

## Monomorphism

$i: X \rightarrow Y$  is a mono if

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ g \downarrow & & \downarrow i \\ X & \xrightarrow{i} & Y \end{array}$$

Whenever this diagram commutes,  
 $f = g$ . "pullback"

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ & \searrow & \downarrow i \\ & & Y \end{array}$$

$$if = ig \iff f = g$$

Mono in Top

Aside: Mono in Set

Should be: injective functions

If  $i$  is not injective, then not  
invariant: Assume  $i(x) = i(y)$ .

$$i: A \rightarrow B$$

$$f, g: \{*\} \rightarrow A, f(*) = x, g(*) = y$$

$$\{*\} \rightarrow A \xrightarrow{i} B$$

$$i(f(*)) = i(x)$$

$$i(g(*)) = i(y)$$

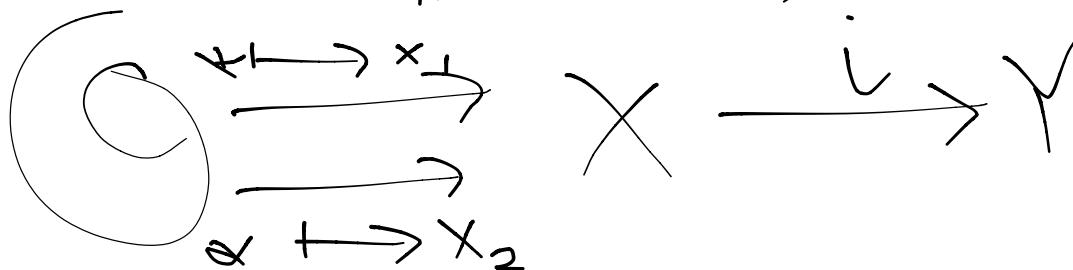
If the underlying function is injective,  
the map of top spaces is a mono.

$$C = (\{*\}, \{\emptyset, \{*\}\})$$

$$f: i: (X, \tau_X) \rightarrow (Y, \tau_Y)$$

$$i \in \underline{\text{Top}}(X, Y)$$

has  $i(x_1) = i(x_2)$



Because  $i(x_1) = i(x_2)$ ,  $i(x \mapsto x_1)$  is the same as  $i(x \mapsto x_2)$ .

Divisible groups ~~X~~

High-level technical condition:

Forgetful  $U \cong \text{Hom}(C, -)$ .

Exact same thing in Ring

Want: mono  $\Rightarrow$  injective

$$\mathbb{Q} = \mathbb{Z}[x]$$

$$\phi: \mathbb{Z}[x] \rightarrow \mathbb{R}$$

$$\begin{array}{ccc}
 1 & \mapsto & 1 \\
 n & \mapsto & \underbrace{1 + \dots + 1}_n
 \end{array}$$

$\text{Hom}(C, -) \cong U$  n times

Let  $\phi: R \rightarrow S$   
with  $\phi(r_1) = \phi(r_2), r_1 \neq r_2$ .

$$\begin{array}{ccc} \mathbb{Z}[x] & \xrightarrow{x \mapsto r_1} & R \\ & \searrow & \downarrow \phi \\ & & S \end{array}$$

$x \mapsto r_2$

$\phi$  is not a mono.

The same argument applies to the category of posets with monotone maps.

$$C = \{+\}$$

Example:  $\mathcal{C} = (\text{l.p.c.}, \text{s.l.s.c. top})$   
psh topos p.c.

(NICE)

Forgetful functor is not of the

from  $\text{Hom}(X, -)$ .

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Epis

1) Epis in Set

Say  $f: A \rightarrow B$  is not a surjection. Say  $b \notin \text{Im} f$ . Let

$$C = \{x_1, x_2\}$$

$$A \xrightarrow{f} B \xrightarrow{\quad} C$$

$g_1(y) = x_1, \forall y \in B$   
 $g_2(y) = \begin{cases} x_1 & y \neq b \\ x_2 & y = b \end{cases}$

2) Epis in Top

$$C = (\{x_1, x_2\}, \{\emptyset, \{x_1, x_2\}\})$$

Property: Every function to  $C$  is automatically continuous.

$f: X \rightarrow Y$  a non-surjective map of topological spaces, let  $b \notin \text{Im } f$ .

$$X \xrightarrow{f} Y \begin{matrix} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{matrix} C$$

$$g_1(y) = x_1, \quad \forall y \in Y$$

$$g_2(y) = \begin{cases} x_1 & y \neq b \\ x_2 & y = b \end{cases}$$

3) Epis in Haus

Non-surjective epis here:

$$i: \mathbb{Q} \rightarrow \mathbb{R} \begin{matrix} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{matrix} X$$

For any  $x \in \mathbb{R}$ ,  $x = \lim_{n \rightarrow \infty} (i(q_n))$

$$\begin{aligned} f(x) &= f\left(\lim_{n \rightarrow \infty} i(q_n)\right) \\ &= \lim_{n \rightarrow \infty} f(i(q_n)) \end{aligned}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} f_2(i(q_n)) \\
&= f_2\left(\lim_{n \rightarrow \infty} i(q_n)\right) \\
&= f_2(x).
\end{aligned}$$


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## Skeletalization

Idea: Force if  $x \cong y$ , then  $x = y$ .

Let  $\mathcal{C}$  a small category.

$\square$ ) Isomorphism is an equivalence relation.

Look at the collection (set) of equivalence classes of objects under isomorphism, and pick one object from each,  $x \in [x]$ .

Consider  $\mathcal{D} =$

- objects are the representatives we pick from each equiv. class
- morphisms are just all morphisms in  $\mathcal{C}$  between our objects in  $\mathcal{D}$ .

↑ "full subcategory"

Have a subcategory in which all non-equal objects are non-isomorphic, and, for every  $y \in \mathcal{C}$ , there is an  $x \in \mathcal{D}$  s.t.  $x \cong y$ .

This is a skeletal subcategory.

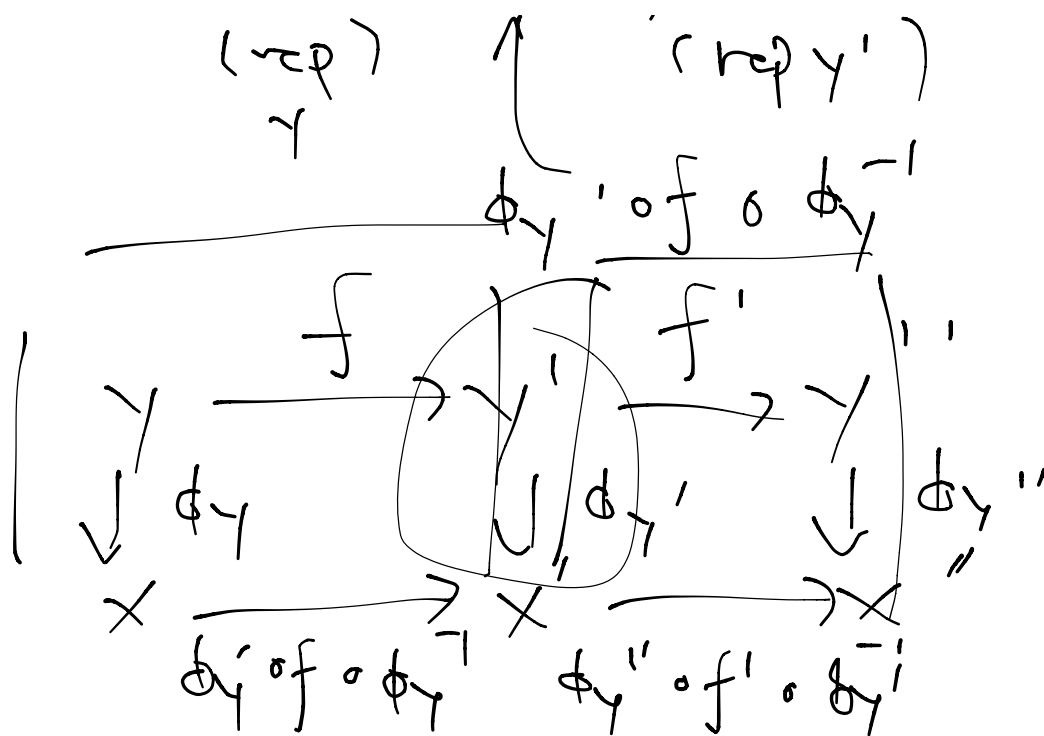
$i: \mathcal{D} \rightarrow \mathcal{C}$  is an equivalence of categories. (There is a functor  $G: \mathcal{C} \rightarrow \mathcal{D}$  s.t.  $Gi \cong \text{id}_{\mathcal{D}}$ ,  $iG \cong \text{id}_{\mathcal{C}}$ ).

$G: \mathcal{C} \rightarrow \mathcal{D}$   
 $y \in \mathcal{C} \mapsto \text{the rep. } x \text{ of } [y]$

Pick, for every  $y \in \mathcal{C}$ , an iso  $\phi_y$  from  $y$  to its representative in  $[y]$ .

If  $f \in \mathcal{C}(y, y')$ , send it to

$$\begin{array}{ccc} y & \xrightarrow{f} & y' \\ \downarrow \phi_y & & \downarrow \phi_{y'} \\ x & \xrightarrow{\quad} & x' \end{array}$$



$$\phi_{Y''} \circ f' \circ \phi_Y^{-1}$$

$$= G(f' \circ f).$$

$$G \circ id = id. \quad (\text{identity on the rep}).$$

$$id \circ G \approx id$$

Define  $\eta: id \rightarrow G$  by:  
for each  $Y \in \mathcal{C}$ ,

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ \downarrow \phi_Y & & \downarrow \phi_{Y'} \\ id(Y) = X & \xrightarrow{\quad} & X' \end{array}$$

$$i_G(f) \quad i_G(y')$$

$$\phi_y' \circ f \circ \phi_y^{-1}$$

Example: FinSet

$$[n] = \{0, 1, \dots, n-1\}$$

The full subcategory on  $\{[n] \mid n \in \mathbb{N}\}$   
is a skeletalization of finite sets.