W Week2

Monomorphism

$i : X \to Y$ is a mono if

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

whenever this diagram commutes, $f = g$ ∴ pullback

$$\begin{tikzcd} Z \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \arrow[r, "i"] & Y \end{tikzcd}$$

$$if = ig \iff f = g$$

Monos in $\underline{\text{Top}}$

Aside: Monos in $\underline{\text{Set}}$

Should be: injective functions

If i is not injective, then not mono: Assume $i(x) = i(y)$.

$i : A \longrightarrow B$

$f, g : \{*\} \longrightarrow A$, $f(*) = x$,
$g(*) = y$

$$\{*\} \rightrightarrows A \xrightarrow{i} B$$

$if(*) = i(x)$
$ig(*) = i(y)$

If the underlying function is injective, the map of top spaces is a mono.

$C = (\{*\}, \{\emptyset, \{*\}\})$

If $i : (X, \tau_X) \longrightarrow (Y, \tau_Y)$
$i \in \underline{\mathrm{Top}}(X, Y)$

has $i(x_1) = i(x_2)$

$$\begin{tikzcd} \bigcirc \arrow[r, shift left, "* \mapsto x_1"] \arrow[r, shift right, "\alpha \mapsto x_2"'] & X \arrow[r, "i"] & Y \end{tikzcd}$$

Because $i(x_1) = i(x_2)$, $i(\alpha \mapsto x_1)$ is the same as $i(* \mapsto x_2)$.

Divisible groups

High-level technical condition:

Forgetful $U \cong \mathrm{Hom}(C, -)$.

Exact same thing in <u>Ring</u>

Want: mono $\Longrightarrow$ injective

$\bigcirc = \mathbb{Z}[x]$.

$$\begin{aligned} \phi : \mathbb{Z}[x] &\longrightarrow R \\ 1 &\longmapsto 1 \\ n &\longmapsto \underbrace{1 + \cdots + 1} \end{aligned}$$

$$\mathrm{Hom}(C, -) \cong U^{\,n \text{ times}}$$

Let $\phi : R \longrightarrow S$
with $\phi(r_1) = \phi(r_2)$, $r_1 \neq r_2$.

$$\begin{tikzcd}
\mathbb{Z}[x] \arrow[r, shift left, "x \mapsto r_1"] \arrow[r, shift right, "x \mapsto r_2"'] & R \arrow[r, "\phi"] & S
\end{tikzcd}$$

ϕ is not a mono.

The same argument applies to the category of posets with monotone maps.

$C = \{*\}$

Example: $\mathcal{C}$ = (l.p.c., s.l.s.c. top
pointed, p.c)

(NICE)

Forgetful functor is not of the

form $\mathrm{Hom}(X, -)$.

Epis

1) Epis in <u>Set</u>

Say $f: A \to B$ is not a surjection. Say $b \notin \mathrm{Im} f$. Let $C = \{x_1, x_2\}$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g_1(y) = x_1 \ \forall y \in B"] \arrow[r, shift right] & C \end{tikzcd}$$

$$g_2(y) = \begin{cases} x_1 & y \neq b \\ x_2 & y = b \end{cases}$$

2) Epis in <u>Top</u>

$$C = (\{x_1, x_2\}, \{\emptyset, \{x_1, x_2\}\})$$

Property: Every function to C is automatically continuous.

$f: X \longrightarrow Y$ a non-surjective map of topological spaces, let $b \notin \operatorname{Im} f$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & C \end{tikzcd}$$

$g_1(y) = x_1 \quad \forall y \in Y$

$$g_2(y) = \begin{cases} x_1 & y \neq b \\ x_2 & y = b \end{cases}$$

3) Epis in $\underline{\text{Haus}}$

Non-surjective epis here:

$$\begin{tikzcd} i: \mathbb{Q} \arrow[r] & \mathbb{R} \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & X \end{tikzcd}$$

For any $x \in \mathbb{R}$, $x = \lim_{n \to \infty} (i(q_n))$

$f_1(x) = f_1(\lim_{n\to\infty} i(q_n))$
$= \lim_{n\to\infty} f_1(i(q_n))$

$$= \lim_{n \to \infty} f_2(i(q_n))$$

$$= f_2\left(\lim_{n\to\infty} i(q_n)\right)$$

$$= f_2(x).$$

Skeletalization

Idea: Force if $x \cong y$, then $x = y$.

Let $\mathcal{C}$ a small category.

I) Isomorphism is an equivalence relation.

Look at the collection (set) of equivalence classes of objects under isomorphism, and pick one object from each, $x \in [x]$.

Consider $\mathcal{D}$ =

- objects are the representatives we picked from each equiv. class
- morphisms are just all morphisms in $\mathcal{C}$ between our objects in $\mathcal{D}$.

"full subcategory"

Have a subcategory in which all non-equal objects are non-isomorphic, and, for every $y \in \mathcal{C}$, there is an $x \in \mathcal{D}$ s.t. $x \cong y$.

This is a skeletal subcategory.

$i : \mathcal{D} \to \mathcal{C}$ is an equivalence of categories. (There is a functor $G : \mathcal{C} \to \mathcal{D}$ s.t. $Gi \cong id_{\mathcal{D}}$, $iG \cong id_{\mathcal{C}}$).

$$G : \mathcal{C} \to \mathcal{D}$$
$$y \in \mathcal{C} \longmapsto \text{the rep. } x \text{ of } [y]$$

Pick, for every $y \in \mathcal{C}$, an iso ϕ_y from y to its representative in $[-]$.

If $f \in \mathcal{C}(y, y')$, send it to

$$\begin{tikzcd}
y \arrow[r, "f"] \arrow[d, "\phi_y"] & y' \arrow[d, "\phi_{y'}"] \\
x \arrow[r] & x'
\end{tikzcd}$$

(rep Y) (rep Y')

$\phi_{Y'} \circ f \circ \phi_Y^{-1}$

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\phi_Y"] & Y' \arrow[r, "f'"] \arrow[d, "\phi_{Y'}"] & Y'' \arrow[d, "\phi_{Y''}"] \\
X \arrow[r, "\phi_{Y'} \circ f \circ \phi_Y^{-1}"'] & X' \arrow[r, "\phi_{Y''} \circ f' \circ \phi_{Y'}^{-1}"'] & X''
\end{tikzcd}$$

$$\phi_{Y''} \circ f' \circ f \circ \phi_Y^{-1}$$

$$= G(f' \circ f).$$

$G\, i = id$. (identity on the rep).

$iG \cong id$.

Define $\eta : iG \to id$ by:
for each $Y \in \mathcal{C}$,

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\phi_Y"] & Y' \arrow[d, "\phi_{Y'}"] \\
iG(Y) = X \arrow[r] & X'
\end{tikzcd}$$

$iG(f) \qquad iG(y')$

$\phi_{y'} \circ f \circ \phi_y^{-1}$

Example: FinSet

$[n] = \{0, 1, \dots, n-1\}$

The full subcategory on $\{[n] \mid n \in \mathbb{N}\}$ is a skeletalization of finite sets.