Adjoints

Def. We say $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ are an adjoint pair (F is left adjoint to G) if there is a natural iso

$$\mathcal{D}(Fx, y) \cong \mathcal{C}(x, Gy)$$

Functors $\mathcal{C} \times \mathcal{D}^{op} \longrightarrow \underline{Set}$.

Ex Vector spaces
Fix a field k. Define $F : \underline{Set} \to \underline{k\text{-Vect}}$ by taking the k-vector space with basis elements of the set.

$$F\{a, b, c\} = \{\underset{\substack{\uparrow \\ \text{finite}}}{}k\text{-linear combos of } a, b, c\}$$

$G : \underline{k\text{-Vect}} \to \underline{Set}$ the forgetful functor: takes the vector space to itself considered as a set.

$f \in Set(x, y)$, then $Ff : Fx \to Fy$ is defined by taking any basis elt in Fx to the basis elt from f: "linearly extend f".

$$\underline{k\text{-Vect}}(F\underset{\nwarrow \text{set}}{S}, V) \cong \underline{\text{Set}}(S, GV)$$

A morphism of vector spaces is determined by where it sends the basis (S).

$$\begin{tikzcd}
S \arrow[r, "\text{canon.}"] \arrow[dr, "f"'] & GFS \arrow[d, dashed, "\exists!\ k\text{-linear map extending } f"] \\
& GV
\end{tikzcd}$$

Ex. $\underset{\uparrow}{\underline{\text{Met}}}$ $\quad$ $\underset{\uparrow}{\underline{\text{CMet}}}$

morphisms are uniformly continuous

If I have a uniformly continuous $f: X \to Y$, Y complete, $\exists!\ \tilde{f}: \tilde{X} \to Y$, where $\tilde{X}$ is the completion of X, s.t.

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr, "f"'] & \tilde{X} \arrow[d, dashed, "\tilde{f} \to \exists!"] \\
& Y
\end{tikzcd}$$

Completion functor F, forgetful G: $\underline{\text{CMet}} \to \text{Me}$

$$\text{Met}(X, GY) \cong \underline{\text{CMet}}(FX, Y)$$

Completion is left adjoint to forgetful.

<u>Lemma</u>. If F is left adjoint to G ($F \dashv G$), then $GF : \mathcal{C} \to \mathcal{C}$, $FG : \mathcal{D} \to \mathcal{D}$.

We get a natural transformation $\eta : 1_{\mathcal{C}} \to GF$, $\varepsilon : FG \to 1_{\mathcal{D}}$, instantiating the iso:

$$\mathcal{D}(Fx, y) \longrightarrow \mathcal{C}(x, Gy)$$

$$f \longmapsto Gf \circ \eta_x$$

$$\begin{tikzcd}
x \arrow[d, "\eta_x"'] \arrow[dr, dashed, "Gf \circ \eta_x"] & \\
GFx \arrow[r, "Gf"] & Gy
\end{tikzcd}$$

$$\mathcal{C}(x, Gy) \longrightarrow \mathcal{D}(Fx, y)$$

$$g \longmapsto \varepsilon_y \circ Fg$$

$$\begin{tikzcd}
Fx \arrow[r, "Fg"] \arrow[dr, dashed, "\varepsilon_y \circ Fg"'] & FGy \arrow[d, "\varepsilon_y"] \\
& y
\end{tikzcd}$$

Proof. For $\eta_x : x \longrightarrow GFx$

$$\mathcal{D}(Fx, Fx) \cong \mathcal{C}(x, GFx).$$
$$id_{Fx} \longrightarrow \eta_x$$

naturality of the iso $\Rightarrow$ η is a natural transformation.

For $\varepsilon_y : FGy \longrightarrow y$

$$\mathcal{C}(Gy, Gy) \cong \mathcal{D}(FGy, y)$$
$$id_{Gy} \longrightarrow \varepsilon_y$$

Ex: Products instantiate isos

$$\mathcal{C}(X, Y_1 \times Y_2) \cong \mathcal{C}^2((X,X), (Y_1, Y_2))$$

"a map $X \longrightarrow Y_1 \times Y_2$ is the same as a pair of maps $X \longrightarrow Y_1$, $X \to Y_2$".

Product is a right adjoint to the diagonal functor

$$1_{\mathcal{C}} \longrightarrow \Pi\Delta$$

$$X \longrightarrow X \times X$$

$$\Delta \Pi \longrightarrow 1_D$$

$$(Y_1 \times Y_2, Y_1 \times Y_2) \longrightarrow (Y_1, Y_2)$$

$$\eta_X : X \longrightarrow X \times X$$

diagonal: induced by $id : X \to X$, $id : X \to X$

ε_{Y_1, Y_2} : projections $\pi_1 : Y_1 \times Y_2 \to Y_1$,
$\pi_2 : Y_1 \times Y_2 \to Y_2$.

$$\begin{tikzcd}
 & X \arrow[dl, bend right, "f_1"'] \arrow[dr, bend left, "f_2"] \arrow[d, dashed, "\exists!"] & \\
 & Y_1 \times Y_2 \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
Y_1 & & Y_2
\end{tikzcd}$$

$$\underbrace{\qquad\qquad\qquad}_{\varepsilon}$$

$$\begin{tikzcd}
(X, X) \arrow[r, "{(f_1, f_2)}"] \arrow[d, "{(\Delta, \Delta)}"] & (Y_1, Y_2) \\
(X \times X, X \times X) \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

$$F \dashv G$$

$$\eta : 1_C \longrightarrow GF$$

$$\varepsilon : FG \longrightarrow 1_D$$

$$\begin{tikzcd}
F = F1_{\mathcal{C}} \arrow[r, "1_F \eta"] \arrow[rrd, bend right] & FGF \arrow[r, "\varepsilon 1_F"] & 1_{\mathcal{D}}F \arrow[d, equal] \\
 & & F
\end{tikzcd}$$

$$\begin{tikzcd}
G = 1_{\mathcal{C}} G \arrow[r, "\overset{1_F}{\eta 1_G}"] \arrow[rr, bend right, "id"'] & GFG \arrow[r, "1_G \varepsilon"] & G1_{\mathcal{D}} = G
\end{tikzcd}$$

$$\left[\begin{tikzcd}
Fx \arrow[r, "F(\eta)"] \arrow[dr, dashed, "F(id_x) = id_{Fx}"'] & FGFx \arrow[d, "\varepsilon_{Fx}"] \\
 & Fx
\end{tikzcd}\right]$$

$$\begin{tikzcd}
x \arrow[r, "\eta"] \arrow[dr, bend right, "\phi"'] & GFx \arrow[d, "\exists ! \psi \text{ st. } G\psi"] \\
 & Gy
\end{tikzcd}$$

$$\begin{tikzcd}
FGy \arrow[r, "\varepsilon"] & y \\
Fx \arrow[u, dashed, "\exists ! \sigma \text{ st } F\sigma"] \arrow[ur, "\tau"'] &
\end{tikzcd}$$

ε_{Fx}

$$\begin{tikzcd}
FGFx \arrow[r, "\cdots"] & Fx \arrow[d, "F(\eta)"] \\
 & FGFx
\end{tikzcd}$$

Wave my hands, paste the triangles together

$$\begin{tikzcd}
Fx \arrow[r, "F(\eta)"] \arrow[dr, "F(id_x) = id"'] & FGFx \arrow[d, "\varepsilon_{Fx}"] \\
 & Fx
\end{tikzcd}$$

$$Hom(Fx, Fx) \cong Hom(x, GFx)$$

η $\underline{\text{is}}$ the image of identity under the isomorphism

$$\begin{tikzcd}
Fx \arrow[r, "F(\eta)"] \arrow[dr, "\varepsilon_{Fx} \circ F(\eta) = id_{Fx}"'] & FGFx \arrow[d, "\varepsilon_{Fx}"] \\
 & Fx
\end{tikzcd}$$

$$\begin{tikzcd}
Gy \arrow[r, "\eta_{Gy}"'] \arrow[dr, dash] & GFGy \arrow[d, "G(\varepsilon_y)"] \\
 & {}
\end{tikzcd}$$

$$id_{Gy} \rightarrow Gy$$

If F, G are adjoints, then it's not necessary for F, G to be inverses (equivalence-inverses). But they do satisfy composing weakly to the identity on their images.

$$G(\varepsilon_y) \circ \eta_{Gy} = id_{Gy}$$

$$\varepsilon_{Fx} \circ F(\eta_x) = id_{Fx}$$

instantiating hom-set iso

Theorem: If $F \dashv G$, $J : I \longrightarrow \mathcal{C}$, and it has a colimit colim J, then

$$F(\operatorname{colim} J) = \operatorname{colim}(F \circ J)$$

$$c : J \longrightarrow \Delta \operatorname{colim} J$$

$$c_x : Jx \longrightarrow \operatorname{colim} J \text{ s.t.}$$

for any $f \in I(x, x')$,

$$Jx \xrightarrow{Jf} Jx'$$

$$\begin{tikzcd} {} \arrow[dr, "\iota_x"'] & & {} \arrow[dl, "\iota_{x'}"] \\ & \operatorname{colim} J & \end{tikzcd}$$

In $\mathcal{D}$, take the object $F(\operatorname{colim} J)$ and the morphisms

$$FJx \xrightarrow[F(\iota_x)]{} F(\operatorname{colim} J)$$

Say we have $Y \in \mathcal{D}$, $\phi : FJ \longrightarrow \Delta Y$

$\phi_x : FJx \longrightarrow Y$

$G\phi_x : GFJx \longrightarrow GY$

$$\begin{tikzcd} Jx \arrow[r, "\eta_{Jx}"] \arrow[d, "\iota_x"'] \arrow[rr, bend left=40, no head] & GFJx \arrow[d, "GF\iota_x"] \arrow[dr, bend left, "G\phi_x"] & \\ \operatorname{colim} J \arrow[r, "\eta_{\operatorname{colim} J}"'] & GF\operatorname{colim} J & GY \end{tikzcd}$$

$G\phi_x \circ \eta_{Jx}$ is a cocone $J \longrightarrow \Delta GY$.

By the property of colim J, ∃!

$\operatorname{colim}(G\phi_x \circ \eta_{Jx}) : \operatorname{colim} J \to Gy$
making the diagram commute.

$$\begin{tikzcd}
\operatorname{colim} J \arrow[rr] \arrow[dr, "\eta_{\operatorname{colim} J}"'] & & Gy \\
& GF \operatorname{colim} J \arrow[ur] &
\end{tikzcd}$$

$F \operatorname{colim} J \to y$

∃! g s.t. Gg makes the diagram commute

$F \operatorname{colim} J$, $F\tau_x : FJx \to F\operatorname{colim} J$
are the colimit.

So left adjoints preserve colimits, similarly, right adjoints preserve limits.

F : <u>Set</u> → <u>kVect</u> free functor

coproducts in <u>Set</u> are disjoint union
coproducts in <u>kVect</u> are direct sum

$$F\left(\coprod_{i\in I} S_i\right) \cong \bigoplus_{i\in I} F(S_i)$$

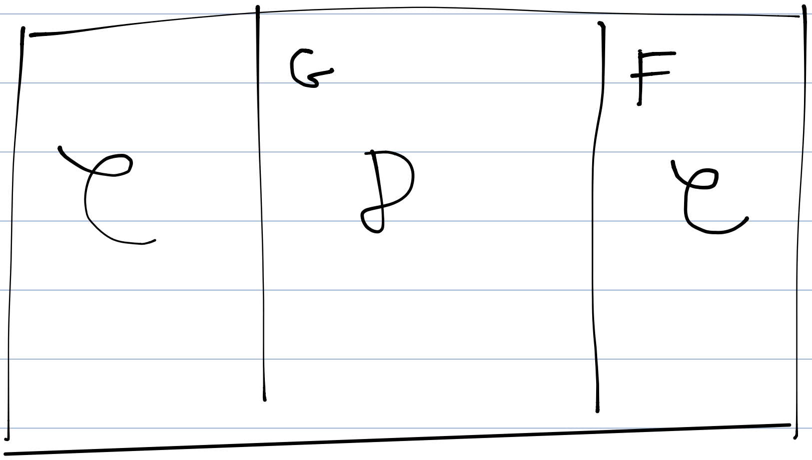
__Ex.__ $\mathcal{C} = \underline{\text{Top}}$, $G : \text{Top} \to \text{Set}$ forgetful.

$H \dashv G \dashv F$

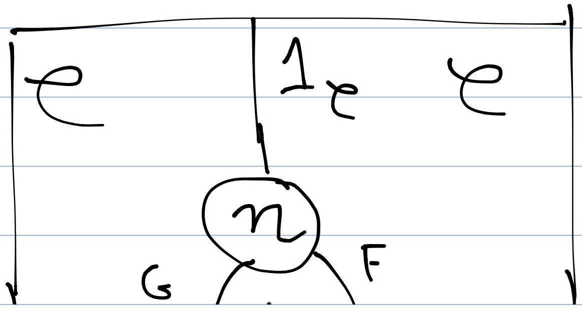
↑ "discrete" ↑ "indiscrete"

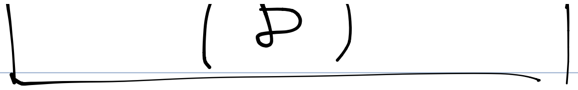
To compute a limit/colimit in $\underline{\text{Top}}$, do it to the underlying set and topologize appropriately.

String diagram:

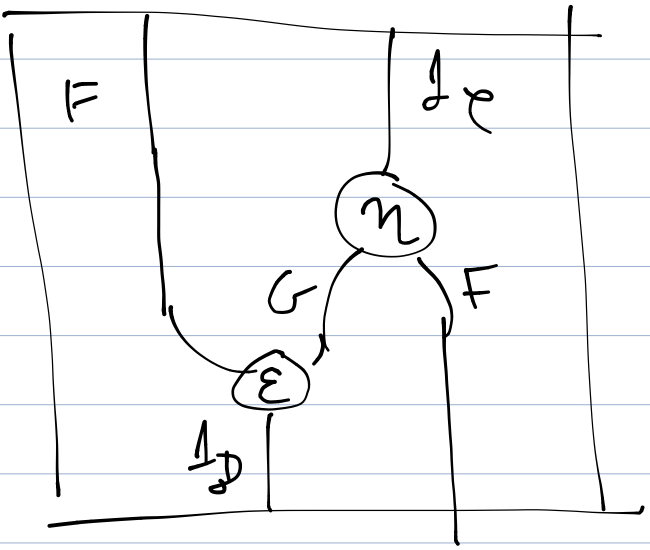


$G \circ F : \mathcal{C} \longrightarrow \mathcal{C}$



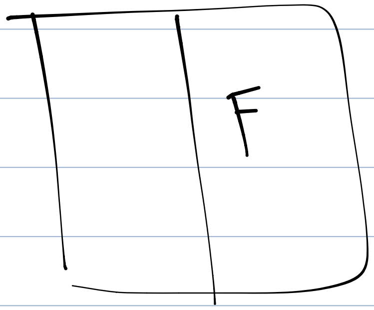


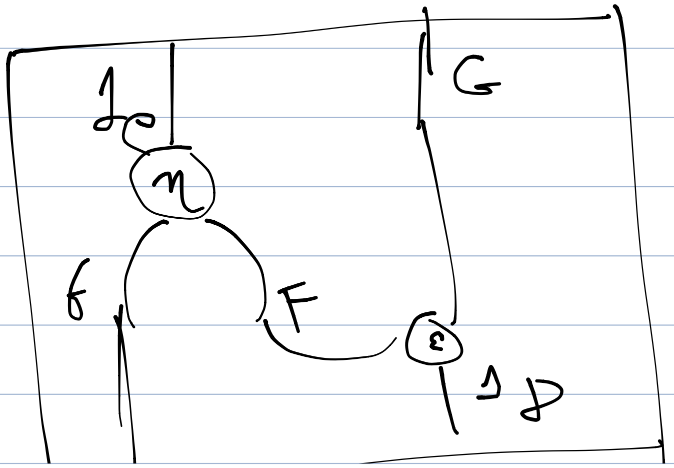
$\eta : 1_{\mathcal{C}} \longrightarrow GF$

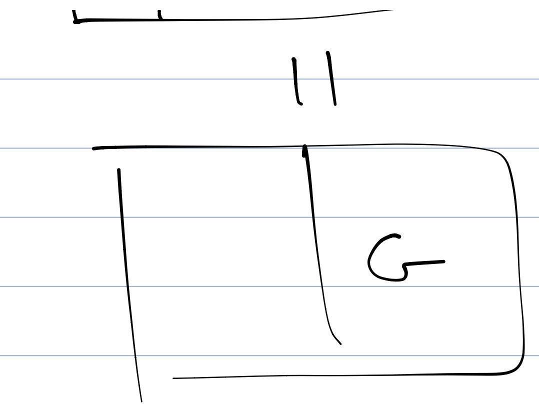


$$\begin{tikzcd}
F \arrow[d, "F\eta"] \\
FGF \arrow[d, "\varepsilon F"] \\
F
\end{tikzcd}$$

||



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