The Yoneda Lemma

Functor categories: Given $\mathcal{C}, \mathcal{D}$, the <u>functor category $\mathcal{C} \to \mathcal{D}$</u> $[\mathcal{C}, \mathcal{D}]$ has

- objects are functors $F: \mathcal{C} \to \mathcal{D}$
- morphisms $F \to G$ are nat trans η
  $\forall x, y \in \mathcal{C}$, $f \in \mathcal{C}(x, y)$

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\eta_x"'] & Fy \arrow[d, "\eta_y"] \\ Gx \arrow[r, "Gf"] & Gy \end{tikzcd}$$

composition is by stacking (pw composition of natural transformations.

Ex. 1: Let $\mathcal{C}$ be the category with one object $\star$, and no non-identity morphisms.

$$\mathcal{C} = \begin{tikzcd} \star \arrow[loop above, "id_\star"] \end{tikzcd}$$

$$[\mathcal{C}, \mathcal{D}] \cong \mathcal{D}$$

$$\begin{tikzcd} F\star \arrow[d, "\mathcal{D}(F\star, G\star)"] \\ G\star \end{tikzcd}$$

Ex. 2 G a group $\longleftrightarrow$ one-object category where all morphisms are invertible

G "delooping" of G

BG

$[BG, \mathcal{D}]$ is the category of $\mathcal{D}$-objects with G-actions (maps $G \to \mathrm{Aut}(d)$).

Ex. 3 "Presheaves on $\mathcal{C}$",
$[\mathcal{C}^{op}, \mathrm{Set}]$.
Category of contravariant functors $\mathcal{C} \to \mathrm{Set}$
(Want $\mathcal{C}$ to be locally small.)

For each $x \in \mathcal{C}$, we have a functor

$h^x = \mathcal{C}(-, x)$.

Objects: any object $y \longmapsto \mathcal{C}(y, x)$.
Morphisms: $f : y \to y'$.

$$y \xrightarrow{f} y' \quad \big\downarrow \phi$$

$\phi \circ f \longrightarrow \check{X}$

$\mathcal{C}(Y', X) \longrightarrow \mathcal{C}(Y, X)$

$\phi \longmapsto \phi \circ f$

- $\mathcal{C}$
- $\mathrm{Psh}\,\mathcal{C} = [\mathcal{C}^{op}, \mathrm{Set}]$
- $X \longmapsto h^X = \mathcal{C}(-, X)$.

The map $X \longrightarrow h^X$ can be extended to a functor $\mathcal{C} \longrightarrow \mathrm{Psh}\,\mathcal{C}$.

Say $f : X \longrightarrow X'$. We want a natural transformation $h^X \longrightarrow h^{X'}$.

$h^X(g) \longrightarrow$ "precompose g"

$$\begin{tikzcd}
\mathcal{C}(Y, X) \arrow[r, "h^X(g)"] \arrow[d, "f \circ -"'] & \mathcal{C}(Y', X) \arrow[d, "f \circ -"] \\
\mathcal{C}(Y, X') \arrow[r, "h^{X'}(g)"'] & \mathcal{C}(Y', X')
\end{tikzcd}$$

$g : Y' \longrightarrow Y$

$\mathcal{C}(-, X) \longrightarrow \mathcal{C}(-, X')$

$\phi : Y \to X \longmapsto f \circ \phi : Y \longrightarrow X'$

Start at the top left: $\phi : Y \longrightarrow X$.

Right then down: $\phi \longmapsto \phi' \circ g \longmapsto f \circ \phi \circ g$

Down, then right: $\phi \longmapsto f \circ \phi \longmapsto f \circ \phi \circ g$.

If $f: X \to X'$, it induces a map $h^X \to h^{X'}$

So we have a functor $\underset{\hat{h}}{\mathcal{C}} \longrightarrow \mathrm{Psh}\,\mathcal{C}$

$$X \longrightarrow h^X$$

$$f: X \to X' \longrightarrow \left[h^X \xrightarrow[f \circ -]{} h^{X'} \right]$$

<u>Theorem</u> (Yoneda) $h: \mathcal{C} \longrightarrow \mathrm{Psh}\,\mathcal{C}$ is full and faithful:

$$\mathrm{Nat}(h^X, h^{X'}) \cong \mathcal{C}(X, X')$$

Surjective   injective

The category $\mathcal{C}$ lives inside the category of functors $\mathrm{Psh}\,\mathcal{C}$.

Lemma: if $F \in \mathrm{Psh}\,\mathcal{C}$, h^X, then

$$\mathrm{Nat}(h^X, F) \cong F(X) \leftarrow \text{a set}$$

$$\eta \longmapsto \eta_X(\mathrm{id}_X)$$

$$\left[\begin{array}{c} h^X(X) = \mathcal{C}(X,X) \\ \downarrow \\ FX \end{array}\right]$$

If we know $\eta_X(\mathrm{id}_X)$, we know all of η:

Pick $X' \in \mathcal{C}$, and we want to know, for any $f : X' \longrightarrow X$, $\eta_{X'}(f)$?

$$\begin{tikzcd}
\mathrm{id}_X \in h^X(X) \arrow[r, "_\circ f'"] \arrow[d, "\eta_X"] & h^X(X') \arrow[d, "\eta_{X'}"] \\
FX \arrow[r, "Ff'"] & FX'
\end{tikzcd}$$

Chase id_X. Right, then down:

$\eta_{X'}(f')$

Down, then right: $(Ff')(\eta_X(\mathrm{id}))$.

$$\eta_{X'}(f') = (Ff')(\eta_X(\mathrm{id}_X)).$$

The map $\eta \longmapsto \underset{\substack{\uparrow \\ FX}}{\eta_X(\mathrm{id}_X)}$ is injective.

To show surjectivity, for any $f : X' \to X$, define
$$\eta_{X'}(f) = (Ff)(\eta_X(\mathrm{id}_X)).$$

For any $g : X' \to X$,

$$\begin{tikzcd} \mathcal{C}(X,X) \arrow[r, "-\circ g"] \arrow[d, "\eta_X"'] & \mathcal{C}(X',X) \arrow[d, "\eta_{X'}"] \\ FX \arrow[r, "Fg"'] & FX' \end{tikzcd}$$

For any $f \in \mathcal{C}(X,X)$,

$$\eta_{X'}(f \circ g) \quad \text{vs} \quad (Fg)(\eta_X(f))$$
$$F(f \circ g)(\eta_X(\mathrm{id}))$$

the same

In particular,
$$\mathrm{Nat}(h^X, h^Y) \cong h^Y(X) = \mathcal{C}(X,Y).$$

Whenever a functor $F : \mathcal{C}^{op} \to \mathrm{Set}$ is isomorphic to a functor h^X, we say F is <u>representable</u>.

Say $F \cong h^X$, $F \cong h^Y$ $\quad h^X \cong h^Y$, $X \cong Y$.

$F:$ <u>Top</u> $\longrightarrow$ Set be the functor which:

Fix two top. spaces Y_1, Y_2. Let

$F(X) = \{ (f,g),\ f: X \to Y_1,\ g: X \to Y_2 \}$

$$\begin{tikzcd}
X' \arrow[r, "\phi"] \arrow[dr, "f\circ\phi"'] \arrow[drr, bend left, "g\circ\phi"] & X \arrow[d, "f"] \arrow[dr, "g"] & \\
& Y_1 & Y_2
\end{tikzcd}$$

$$\begin{tikzcd}
& X \arrow[dl, bend right, "f"'] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left, "g"] & \\
Y_1 & Y_1 \times Y_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y_2
\end{tikzcd}$$

$F \cong h^{Y_1 \times Y_2}$. The statement "the product is unique up to isomorphism"

$$\begin{tikzcd}
& X \arrow[dl, bend right] \arrow[d, dashed, "\exists!"] \arrow[dr, bend left] & \\
Y_1 & Z \arrow[l] \arrow[r] & Y_2
\end{tikzcd}$$

then $Z \cong Y_1 \times Y_2$. Because
$F \cong h^Z$.

<u>Big idea</u> Representable functors:
track the identity.

$$\mathcal{C} \xrightarrow{h} \mathrm{Psh}\,\mathcal{C}$$

$$\mathcal{D} \xrightarrow{i} \mathrm{Psh}\,\mathcal{D}$$

If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor, then
it gives a map $\mathcal{C}(x,y) \longrightarrow$
$\mathcal{D}(Fx, Fy)$.

$$\mathcal{C}(-,y) \longrightarrow \mathcal{D}(-,Fy) :$$

Not every object in $\mathcal{D}$ is in the image
of F.

The Yoneda embedding h has
the following properties.

1. It preserves all (small limits).

$$h(X \times Y) \cong h(X) \times h(Y)$$

2. It is the free cocompletion of $\mathcal{C}$.
(a sort of 2-adjunction).

$$\begin{tikzcd}
 & \mathrm{Psh}(\mathcal{C}) \arrow[dr, "\exists !\ \text{cocontinuous}"] & \\
\mathcal{C} \arrow[ur, "h"] \arrow[rr, "F"'] & & \mathcal{D}
\end{tikzcd}$$

(cocomplete)

Examples of functor categories

$$[\{*\}, \mathcal{D}] \cong \mathcal{D}$$

A representation of a (group, ring, whatever) is an appropriate map of an object associated to each element of the representing object.

In G-Rep, objects are (V, ρ)
$\rho : G \longrightarrow \mathrm{Aut}\, V$
A morphism $(V,\rho) \to (W,\sigma)$ is a linear map $V \to W$ which preserves G-structure.

$\forall g \in G$

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\phi"'] & V \arrow[d, "\phi"] \\ W \arrow[r, "\sigma(g)"] & W \end{tikzcd}$$

This is a naturality square! The functor is taken by (V, ρ) and (W, σ)

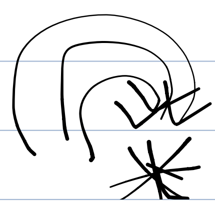
$BG =$ 

(V, ρ) is the functor $* \to V$, $g \to \rho(g)$

(W, σ) " " $* \to W$, $g \to \sigma(g)$

An $R-S$ bimodule is normally thought of as an abelian group with a commuting left and right R- and S- action.

It's an R-module object in Mod-S:

R is  R worth

R is a one-object additive category, and the category of R–S bimodules is the category of additive functors

$$\begin{tikzcd} * \arrow[loop left, "R"] \arrow[r] & \text{Mod-}S. \end{tikzcd}$$

Example. Let $\mathcal{C}$

$$\begin{tikzcd} 1 \arrow[loop above, "id_1"] \arrow[r] & 2 \arrow[loop above, "id_2"] \end{tikzcd}$$

$[\mathcal{C}, \mathcal{D}]$:

Objects are morphisms in $\mathcal{D}$:

$$F(1) \xrightarrow{F(\longrightarrow)} F(2)$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g_1"] & Y \arrow[d, "g_2"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

Slice Category

Let $\mathcal{C}$ be a category, $X \in \mathcal{C}$.

Objects in $\mathcal{C}\downarrow X$ are: pairs (Y, f)

$$\begin{tikzcd} Y \arrow[d, "f"] \\ X \end{tikzcd}$$

A morphism $(Y,f) \longrightarrow (Z,g)$ is a map $\phi \in \mathcal{C}(Y,X)$ s.t.

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

Typical example: category of k-schemes.

Coslice category:

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, "\phi"] & & Z \end{tikzcd}$$

$Y \quad Y \qquad \to Z$

Typical example: $\mathcal{C} = \underline{\mathrm{Top}}$, $x = \{*\}$.

This is $\underline{\mathrm{Top}_*}$, pointed top. spaces.

(Y, y_0), (Z, z_0)

$$\begin{tikzcd} & \{*\} \arrow[dl, "\{y_0\}"'] \arrow[dr, "z_0"] & \\ Y \arrow[rr, "\phi"] & & Z \end{tikzcd}$$

$\phi(y_0) = z_0$.

Nice thing: "Density theorem": every $F: \mathcal{C}^{op} \to \mathrm{Set}$ is a limit of representable functors.
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