$\mathcal{C}$ = Sets with G-actions (G-sets)

$$\begin{tikzcd} X \arrow[r, "\rho(g)"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "\sigma(g)"] & Y \end{tikzcd} \qquad \begin{array}{l} (X, \rho) \\ (Y, \sigma) \end{array}$$

Objects : Functors $BG \longrightarrow \underline{Set}$
Morphisms : Natural transformations

<u>Tannaka Reconstruction Theorem</u>
G can be recovered from the category G-Set and the forgetful functor $U: G\text{-Set} \longrightarrow \underline{Set}$.

$$G \cong Nat(U, U)$$

BG is the category with object $*$ and $BG(*, *)$ are exactly G.

$$\begin{aligned} F: BG &\longrightarrow Set \\ * &\longmapsto X \\ g &\longmapsto \rho(g) : X \longrightarrow X \end{aligned}$$

Claim: $U \cong Hom((G, \text{left action}), -)$.

Natural $\eta : U \longrightarrow \mathrm{Hom}_{G\text{-Set}}((G, H), -)$.

$\eta_{(X,\rho)} : U((X,\rho)) \longrightarrow \mathrm{Hom}_{G\text{-Set}}(((G,H), (X,\rho))$

$X \ni x \longrightarrow (1 \longrightarrow x)$

elements of $X \longleftrightarrow$ G-set maps $(G, H) \longrightarrow X$

1. A morphism ϕ $(G,H) \longrightarrow (X,\rho)$ is uniquely det. by where it sends 1:

$g = g \cdot 1 \longmapsto \rho(g) \cdot \phi(1)$

2. Given any $x \in X$, there is such a map

$\phi_x : (G,H) \longrightarrow (X,\rho)$.

$g \longmapsto \rho(g) \cdot x$

Have to check naturality: $(X,\rho), (Y,\sigma)$

$\phi : (X,\rho) \longrightarrow (Y,\sigma)$

$$\begin{tikzcd}
x \in U(X,\rho) \arrow[r, "U(\phi)"] \arrow[d, "\eta"'] & U(Y,\sigma) \arrow[d, "\eta"] \\
G\text{-Set}((G,H),(X,\rho)) \arrow[r, "\phi\circ -"] & G\text{-Set}((G,H),(Y,\sigma))
\end{tikzcd}$$

Right, then down:

$x \longmapsto \phi(x) \longmapsto$ the unique map $1 \to \phi(x)$

Down, then right.

$x \longmapsto$ the unique map $1 \mapsto x \longmapsto 1 \mapsto \phi(x)$

$U \cong \underline{G\text{-Set}}\,((G,+), -)$.

$\mathrm{Nat}(U, U) \cong G\text{-Set}\,((G,+), (G,+))$

Setwise, this is just $U((G,+)) = G$.

$$G \longrightarrow \mathrm{Nat}(U,U)$$
$$g \longmapsto (\eta_{(X,\rho)} = \rho(g))$$

$G \cong \mathrm{Nat}(U,U)$.

$G \not\cong \underbrace{\mathrm{Nat}(\mathrm{id}, \mathrm{id})}_{\text{elements of this should be } G\text{-maps}} \cong Z(G)$

$$\begin{tikzcd}
(X,\rho) \arrow[r, "\phi"] \arrow[d, "\eta_{(X,\rho)}"'] & (Y,\sigma) \arrow[d, "\eta_{(Y,\sigma)}"] \\
(X,\rho) \arrow[r, "\phi"] & (Y,\sigma)
\end{tikzcd}$$

should be
G-maps

Tannaka for Rings

R-Mod is the additive functor category

$$(\circlearrowright\!\ast)^R = BR \longrightarrow Ab$$

Ab-enriched functors $BR \to Ab$

$$F(r+s) = F(r) + F(s)$$

$R \curvearrowright M$ then $(r+s)\cdot m = r\cdot m + s\cdot m$

Let $U : R\text{-Mod} \to Ab$ be the forgetful functor. Then $R \cong \mathrm{Nat}(U,U)$.

$$r \longmapsto (\eta_M(x) = r\cdot x).$$

$$U \cong \mathrm{Hom}_R(R, -).$$

(A map $R \to M$ is uniquely determined by where it sends 1, and for every element there is a map $1 \to M$).

$$\mathrm{Hom}_R({}_RR, {}_RR) \cong R.$$

$$\begin{tikzcd}
1 \arrow[d] & R \arrow[r, "x \mapsto sx"] \arrow[d] & R \arrow[d] & 1 \arrow[d] \\
r & R \arrow[r, "x \mapsto sx"'] & R & r
\end{tikzcd}$$
right mult by r

$$sr = sr.$$

$$U = \mathrm{Hom}_R({}_RR, -).$$
$$\mathrm{Nat}(U, U) \cong \mathrm{Hom}_R({}_RR, {}_RR).$$

Non-Tannaka
$\mathcal{C} = k\text{-Vect}$

$$U : \mathcal{C} \longrightarrow \mathrm{Ab}$$
$$V \longmapsto V \otimes k^2$$
$$U = - \otimes k^2$$

U is additive, faithful: $f, g : V \longrightarrow W$,
$f \otimes id_{k^2} = g \otimes id_{k^2} \iff f = g$.

$$\mathrm{Nat}(U, U) = M_2(k).$$

The categories k-Vect and $M_2(k)$-Mod are equivalent. For any n, there is an additive faithful functor $F_n : k\text{-Vect} \to$ Ab s.t.

$$\mathrm{Nat}(F_n, F_n) \cong M_n(k).$$

$$\mathrm{Nat}(id, id) \cong Z(R),$$

$$Z(M_2(k)) \cong k.$$

$$Z(M_n(k)) \cong Z(M_m(k)).$$

$$G_1 = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad G_2 = \mathbb{Z}_4.$$

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{C}}(G_1), & & \mathrm{Rep}_{\mathbb{C}}(G_2). \\ \downarrow & & \downarrow \\ [BG_1, \mathbb{C}\text{-Vect}] & & [BG_2, \mathbb{C}\text{-Vect}] \end{array}$$

"$\mathbb{C}$-linear representations"

$$\begin{array}{c} \mathrm{Rep}_{\mathbb{C}}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathrm{Rep}_{\mathbb{C}}(\mathbb{Z}_4). \\ \cong \\ \mathbb{C}\mathrm{Vect} \times \mathbb{C}\mathrm{Vect} \times \mathbb{C}\mathrm{Vect} \times \mathbb{C}\mathrm{Vect} \end{array}$$

$\mathbb{C}\mathrm{Vect} \times \mathbb{C}\mathrm{Vect} \times$ ...

$\mathbb{C}\mathrm{Vect}^4$.

$$\begin{tikzcd} G \arrow[r, "i"] \arrow[dr] & \mathbb{C}G \arrow[d, dashed, "\exists!"] \\ & \mathrm{End}_{\mathbb{C}}(V) \end{tikzcd}$$

$\mathbb{C}G = \{ \sum_{\text{finite}} a_g g \mid g \in G \}$

$g \cdot h = gh$, extend linearly

$\mathrm{Rep}_{\mathbb{C}}(G) \cong \mathbb{C}G\text{-Mod}$.

$\mathbb{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $\mathbb{C}(\mathbb{Z}_4) \cong \mathbb{C}^4$.

What if there's no rep. object for the forgetful?

$\mathcal{C}$ is unitary reps of $G = \mathbb{Z}$:

$1 \longmapsto$ an operator on a Hilbert space $\mathcal{H}$ with $T^*T = TT^* = 1$.

$\mathrm{Nat}(u, u) \cong \hat{\mathbb{Z}}$, profinite completion.