Differential Equations

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Course Organization

Grading:

• Homework	30%
• Midterm I Exam	20%
• Midterm II Exam	20%
• Final Exam	30%

Homework assignments

- assigned weekly
- posted on class website
- encouraged to discuss with your classmates
- work you turn in **must be your own work**

Course Website:

- Gauchospace website.
- syllabus
- homework assignments (WebWorks)
- supplemental materials

Any questions?

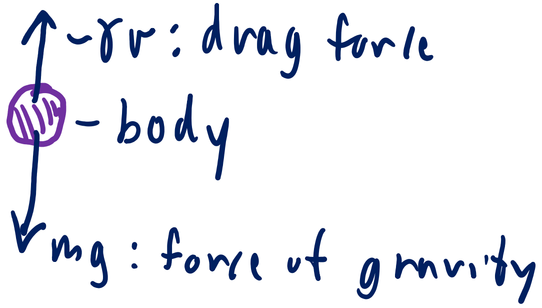
Introduction

Differential Equations give relationships for rates of change.

Ex: $\frac{dy}{dt} = f(t,y)$, $f \leftarrow$ rate function

Ex: Mechanics: Newton's Second Law $F = ma$.

$a = \frac{dv}{dt}$, v: velocity, $m\frac{dv}{dt} = F$



$$m\frac{dv}{dt} = -\gamma v + mg.$$

units: mass ~ kg, time ~ s, dist ~ m

$F \sim kg \cdot m/s^2$, $\gamma \sim kg/s$

$g = 9.8\ m/s^2$

Ex: $m = 10$ kg, $\gamma = 2$ kg/s

$$m\frac{dv}{dt} = -\gamma v + mg$$

- $\frac{dv}{dt} = 9.8 - \frac{v}{5}$, $v(t)$
- initial conditions $v(0) = 50$, $\frac{dv(0)}{dt} = -0.2$
- instead, we had $v(0) = 49$, $\frac{dv(0)}{dt} = 0$

SIR Model

$$R_0 = \frac{\beta N}{\gamma}$$

Ex: (SIR Model)

$$S \xrightarrow{\beta I S} I \xrightarrow{\gamma I} R$$

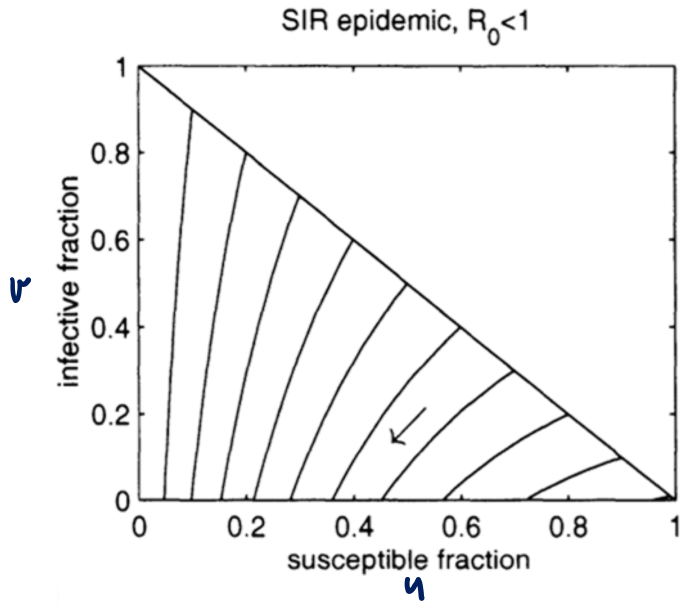
$S(t)$: Susceptible, $I(t)$: Infected, $R(t)$: Recovered

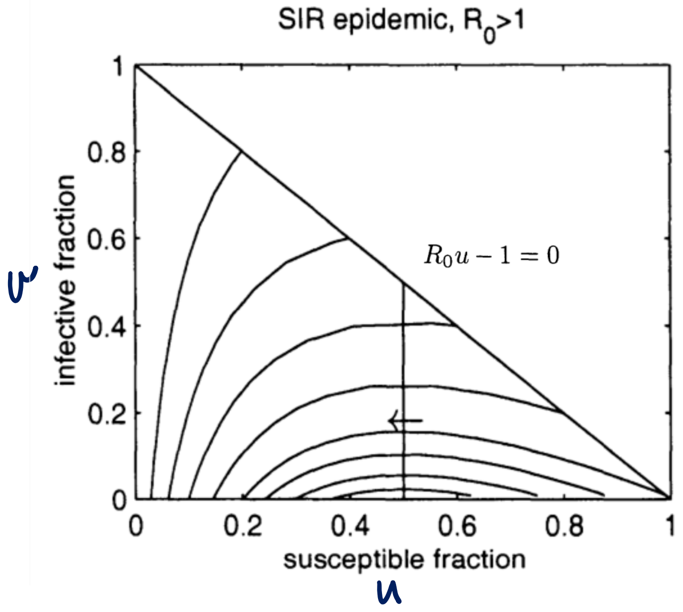
$$\frac{dS}{dt} = -\beta I S, \quad \frac{dI}{dt} = \beta I S - \gamma I, \quad \frac{dR}{dt} = \gamma I$$

Notice: $N(t) = S(t) + I(t) + R(t)$, $\dfrac{dN}{dt} = 0$, $N(t) = N(0)$, $\forall t$.

$$u = \frac{S}{N}, \quad v = \frac{I}{N}, \quad w = \frac{R}{N}, \quad R_0 = \frac{\beta N}{\gamma}, \quad \tau = \gamma t, \quad u + v + w = 1.$$

$$\frac{du}{d\tau} = -R_0 u v, \quad \frac{dv}{d\tau} = (R_0 u - 1) v, \quad \frac{dw}{d\tau} = v.$$





Classifying Differential Equations

Broad Types: (i) Ordinary Differential Equations (ODEs)

$$F\left[t, y, \frac{dy}{dt}, \ldots, \frac{d^n y}{dt^n}\right] = 0, \quad y(t)$$

- only derivatives in one variable
- order is largest derivative that appears.

(ii) Partial Differential Equations (PDEs)

$$G\left[t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t \partial x}, \ldots, \frac{\partial^n u}{\partial t \partial x^{n-1}}, \frac{\partial^n u}{\partial x^n}\right] = 0, \quad u(x,t).$$

- partial derivatives in multiple variable appear.
- order is the largest derivative that appears.

Classifying Differential Equations

- Linear vs Non-linear

  (i) Linear DE's

  $$F\left[t, y, \frac{dy}{dt}, \ldots, \frac{d^n y}{dt^n}\right] = 0. \quad F \text{ is linear in } \frac{d^k y}{dt^k};\ k = 0, 1, \ldots, n$$

  $$\left(y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \ldots, \frac{d^n y}{dt}\right).$$

  $$G\left[t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^n u}{\partial x^n}, \frac{\partial^m u}{\partial t^m}\right] = 0, \quad G \text{ is linear } \frac{\partial^\ell u}{\partial x^j \partial t^k},\ k + j = \ell \leq n.$$

  (ii) Non-linear DE's

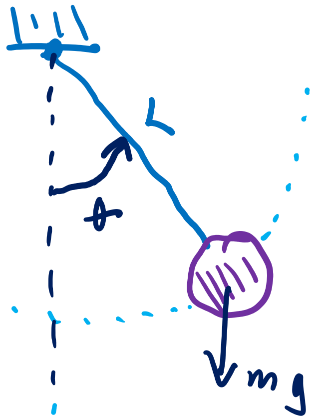
  (not linear)

- Autonomous vs Non-autonomous (i) $\frac{dy}{dt} = g(y)$, (ii) $\frac{dy}{dt} = f(t, y)$, direct dependence on time.
- Homogeneous vs Non-homogeneous (i) $a(x)\frac{dy}{dx} + b(x)y = 0$, (ii) $a(x)\frac{dy}{dx} + b(x)y = g(x)$
- Separable vs non-separable (i) $\frac{dy}{dx} = f(x)g(y)$ (ii) not able to factor into $f(x)g(y)$.

  $$\frac{1}{g(y)}dy - f(x)\,dx = 0$$

Classifying Differential Equations

Ex: Pendulum and Mechanics

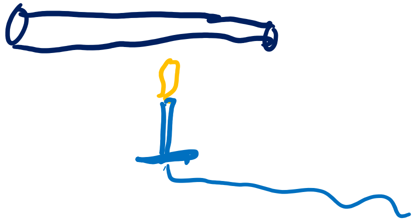


$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta)$$

Non-linear ODE, Homogeneous, Autonomous.

Ex: Heat Equation

$$\frac{\partial u}{\partial t} = K(x)\frac{\partial^2 u}{\partial x^2} + g(x)$$



Linear PDE, Inhomogeneous

Ex: Linearized Pendulum $\sin(\theta) \approx \theta$, $|\theta| \ll 1$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

Linear ODE, Homogeneous, Autonomous.