Differential Equations

Paul J. Atzberger
Department of Mathematics
University of California Santa Barbara

Integral Curves

Direction Fields

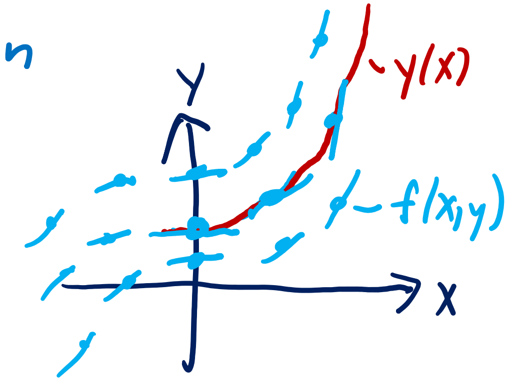
Direction Fields and Integral Curves

$\frac{dy}{dx} = f(x,y)$, $f \leftarrow$ rate function

Geometric Interpretation

$\frac{dy}{dx}$ = slope at $(x,y) = f(x,y)$

$y(x)$ is solution of $\frac{dy}{dx} = f(x,y)$ matches slope $f(x,y)$ at each (x,y)



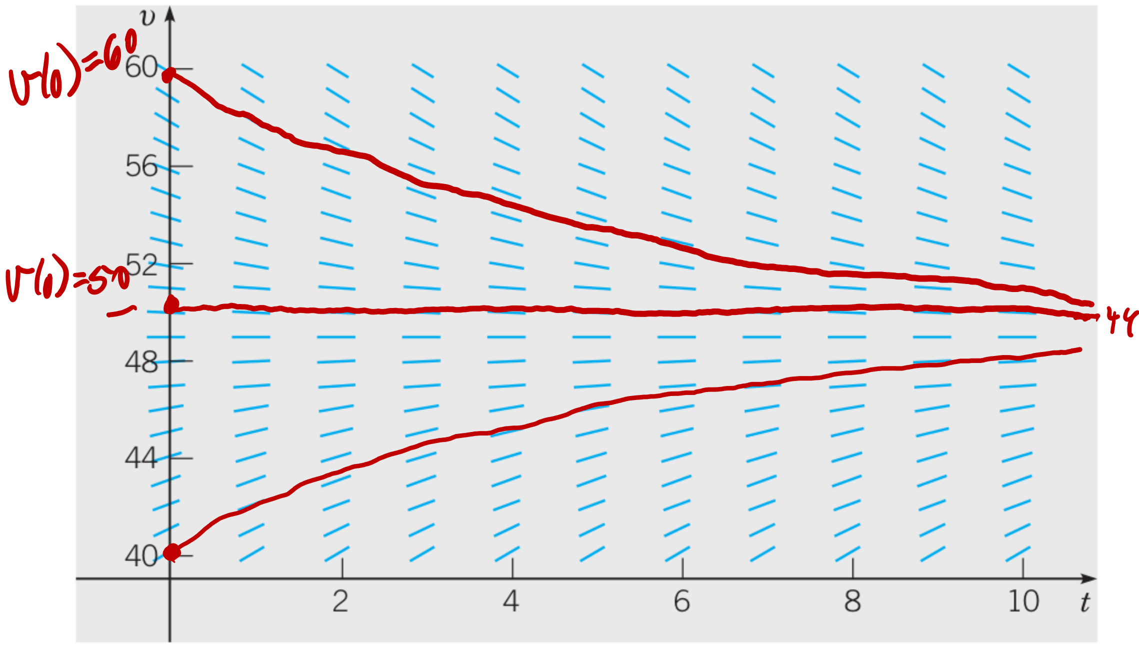
The collection of slopes given by $f(t,y)$ is called the direction field.

Any curve that has tangent slope matching $f(t,y)$ is called an integral curve.

Ex: $m = 10\ \text{kg}$, $\gamma = 2\ \text{kg/s}$

$$m\frac{dv}{dt} = -\gamma v + mg$$

- $\frac{dv}{dt} = 9.8 - \frac{v}{5}$, $v(t)$
- initial conditions $v(0) = 50$, $\frac{dv(0)}{dt} = -0.2$
- instead, we had $v(0) = 49$, $\frac{dv(0)}{dt} = 0$



Integrating Factors

Integrating Factors

$\frac{dy}{dt} = f(t,y)$

Specialize to cases $f(t,y) = -p(t)y + g(t)$

$\frac{dy}{dt} + p(t)y = g(t)$

This can be transformed using integrating factor $\mu(t)$.

$\mu(t)\frac{dy}{dt} + \mu(t)p(t)\,y = \mu(t)g(t).$

Find $\mu(t)$ so that

$\frac{d}{dt}(\mu(t)y) = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y = \mu(t)\frac{dy}{dt} + (\mu(t)p(t))y$

This holds provided

$\frac{d\mu}{dt} = \mu(t)p(t) \overset{\mu(t)\neq 0}{\Rightarrow} \frac{\frac{d\mu}{dt}}{\mu(t)} = \frac{d}{dt}\ln(\mu(t)) = p(t)$

$\Rightarrow \ln(\mu(t)) = \int_{t_0}^{t} p(s)\,ds + c_0$

$\Rightarrow \mu(t) = c_1 e^{\int_{t_0}^{t} p(s)\,ds}$, let $c_1 = 1$

$\mu(t) = e^{\int_{t_0}^{t} p(s)\,ds}$. (integrating factor)

This provides the $\mu(t)$ so that

$\mu(t)\frac{dy}{dt} + \mu(t)p(t) = \frac{d}{dt}(\mu(t)y).$

This gives for ODE

$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t)$

$\Rightarrow \mu(t)y(t) = \int_{t_0}^{t} \mu(s)g(s)\,ds + c_0$

$$\Rightarrow \boxed{\begin{gathered} y(t) = \frac{1}{\mu(t)}\int_{t_0}^{t} \mu(s)g(s)\,ds + \frac{c_0}{\mu(t)} \\ \mu(t) = e^{\int_{t_0}^{t} p(s)\,ds} \end{gathered}}$$

Integrating Factors

Ex: $\frac{dy}{dt} = -ay + b, \quad a \neq 0.$

$$\frac{dy}{dt} + ay = b$$

$$\mu(t)\left(\frac{dy}{dt} + ay\right) = \mu(t)\frac{dy}{dt} + \mu(t)ay = \mu(t)b, \qquad \frac{d\mu(t)}{dt} = a\mu(t)$$

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y \qquad \Rightarrow \mu(t) = e^{ta}$$

$$\frac{d}{dt}\left(e^{at}y\right) = e^{at}b \Rightarrow \int_{t_0}^{t}\frac{d}{ds}\left(e^{as}y\right)ds = \int_{t_0}^{t}e^{as}b\,ds$$

$$\Rightarrow e^{at}y(t) - e^{at_0}y(t_0) = b\,\frac{1}{a}\left[e^{at} - e^{at_0}\right]$$

$$\Rightarrow y(t) = e^{-(t-t_0)a}y(t_0) + \frac{b}{a}\left[1 - e^{-(t-t_0)a}\right].$$

Integrating Factors

Ex! $\left\{\begin{array}{l} t\frac{dy}{dt}+2y=4t^2, \quad t>0 \\ y(1)=2 \end{array}\right\}$

$$\frac{dy}{dt}+p(t)y=g(t)$$

$$\frac{dy}{dt}+\frac{2}{t}y=4t$$

$$\frac{d}{dt}(\mu(t)y)=\mu(t)4t, \quad \frac{d\mu}{dt}=\frac{2}{t}\mu(t) \Rightarrow \mu(t)=e^{\int\frac{2}{t}dt}=e^{2\ln|t|}=t^2$$

$$\frac{d}{dt}(t^2y)=4t^3 \Rightarrow t^2y(t)-t_0^2y(t_0)=\int_{t_0}^{t}4t^3\,ds=[t^4-t_0^4]$$

$$\Rightarrow y(t)=\frac{t_0^2}{t^2}y(t_0)+\frac{(t^4-t_0^4)}{t^2}$$

$t_0=1, y(t_0)=2$

Solution $y(1)=2$ $\Rightarrow$ $y(t)=\frac{2}{t^2}+t^2-\frac{1}{t^2}=\frac{1}{t^2}+t^2.$

Separation of Variables

Separation of Variables

$$\frac{dy}{dx} = f(x,y)$$

Consider the specialized case when
$$f(x,y) = -\frac{M(x)}{N(y)}.$$

We call such ODE's separable.

Remark: $M(\cdot)$ only depends on x,
$N(\cdot)$ only depends on y.

General Method

$$\frac{dy}{dx} = f(x,y) = -\frac{M(x)}{N(y)} \Rightarrow N(y)\,dy = -M(x)\,dx$$
$$\Rightarrow M(x)\,dx + N(y)\,dy = 0$$

This can be used to obtain an implicit equation for $y(x)$.

Consider $\tilde{x} = \tilde{x}(s)$, $\tilde{y} = \tilde{y}(s)$

$$M(\tilde{x})\frac{d\tilde{x}}{ds} + N(\tilde{y})\frac{d\tilde{y}}{ds} = 0$$

$$\int_{s_0}^{s} M(\tilde{x})\frac{d\tilde{x}}{d\tilde{s}}\,d\tilde{s} + \int_{s_0}^{s} N(\tilde{y})\frac{d\tilde{y}}{d\tilde{s}}\,d\tilde{s} = 0$$

$$d\tilde{x} = \frac{d\tilde{x}}{d\tilde{s}}\,d\tilde{s}, \quad d\tilde{y} = \frac{d\tilde{y}}{d\tilde{s}}\,d\tilde{s}$$

$$\tilde{x}(s_0) = x_0, \quad \tilde{x}(s) = x$$
$$\tilde{y}(s_0) = y_0, \quad \tilde{y}(s) = y$$

$$\int_{x_0}^{x} M(\tilde{x})\,d\tilde{x} + \int_{y_0}^{y} N(\tilde{y})\,d\tilde{y} = 0$$

This gives relationship between y and x.

This can be used to determine $y(x)$.