Differential Equations

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Separation of Variables

Separation of Variables

$$\frac{dy}{dx} = f(x,y)$$

Consider the specialized case when

$$f(x,y) = -\frac{M(x)}{N(y)}.$$

We call such ODE's separable.

Remark: $M(\cdot)$ only depends on x,
$N(\cdot)$ only depends on y.

General Method

$$\frac{dy}{dx} = f(x,y) = -\frac{M(x)}{N(y)} \Rightarrow N(y)\,dy = -M(x)\,dx$$

$$\Rightarrow M(x)\,dx + N(y)\,dy = 0$$

This can be used to obtain an implicit equation for $y(x)$.

Consider $\tilde{x} = \tilde{x}(s)$, $\tilde{y} = \tilde{y}(s)$

$$M(\tilde{x})\frac{d\tilde{x}}{ds} + N(\tilde{y})\frac{d\tilde{y}}{ds} = 0$$

$$\int_{s_0}^{s} M(\tilde{x})\frac{d\tilde{x}}{d\tilde{s}}\,d\tilde{s} + \int_{s_0}^{s} N(\tilde{y})\frac{d\tilde{y}}{d\tilde{s}}\,d\tilde{s} = 0$$

$$d\tilde{x} = \frac{d\tilde{x}}{d\tilde{s}}\,d\tilde{s}, \quad d\tilde{y} = \frac{d\tilde{y}}{d\tilde{s}}\,d\tilde{s}$$

$$\tilde{x}(s_0) = x_0, \quad \tilde{x}(s) = x$$

$$\tilde{y}(s_0) = y_0, \quad \tilde{y}(s) = y$$

$$G(x,y(x)) = \int_{x_0}^{x} M(\tilde{x})\,d\tilde{x} + \int_{y_0}^{y} N(\tilde{y})\,d\tilde{y} = 0$$

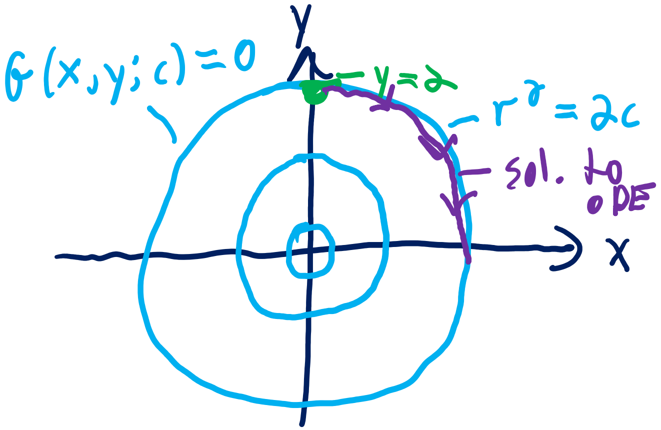
This gives relationship between y and x.

This can be used to determine $y(x)$.

Separation of Variables

Ex: $\begin{cases} \frac{dy}{dx} = \frac{-x}{y} \\ y(0) = 2 \end{cases}$ ← initial condition

, separable $f(x,y) = \frac{-M(x)}{N(y)} = \frac{-x}{y}$, $M(x) = x$, $N(y) = y$.



$G(x, y(x)) = 0, \quad G(x,y) = \int_{x_0}^{x} M(\tilde{x})\,d\tilde{x} + \int_{y_0}^{y} N(\tilde{y})\,d\tilde{y}$

$= \int_{x_0}^{x} \tilde{x}\,d\tilde{x} + \int_{y_0}^{y} \tilde{y}\,d\tilde{y} = \left[\frac{1}{2}\tilde{x}^2\right]_{x_0}^{x} + \left[\frac{1}{2}\tilde{y}^2\right]_{y_0}^{y}$

$= \frac{1}{2}(x^2 + y^2) - \frac{1}{2}(x_0^2 + y_0^2), \quad c = \frac{1}{2}(x_0^2 + y_0^2)$

$G(x,y) = 0 \Rightarrow \frac{1}{2}(x^2+y^2) = \frac{1}{2}(x_0^2 + y_0^2) = c,$

$\Rightarrow \frac{1}{2}(x^2+y^2) = c$

$\frac{dy}{dx} = \frac{-x}{y} \Rightarrow y\,dy = -x\,dx$

$\Rightarrow \int y\,dy = \int -x\,dx + C$

$\Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$

$\Rightarrow \frac{1}{2}(x^2+y^2) = c$

Solution $y(x) = \pm\sqrt{2c - x^2}$

Making use of the initial condition $x=0, y=2$

$y(0) = \pm\sqrt{2c - 0^2} = 2 \Rightarrow y(0) = +\sqrt{2c} = 2 \Rightarrow 2c = 4 \Rightarrow c = 2.$

$\Rightarrow y(x) = \sqrt{4 - x^2}$

Separation of Variables: Direction Field

Ex: $\frac{dy}{dx} = \frac{x^2}{1-y^2}$

$(1-y^2)dy = x^2dx \Rightarrow x^2dx - (1-y^2)dy = 0$

$\int x^2dx - \int(1-y^2)\,dy = \tilde{c}$

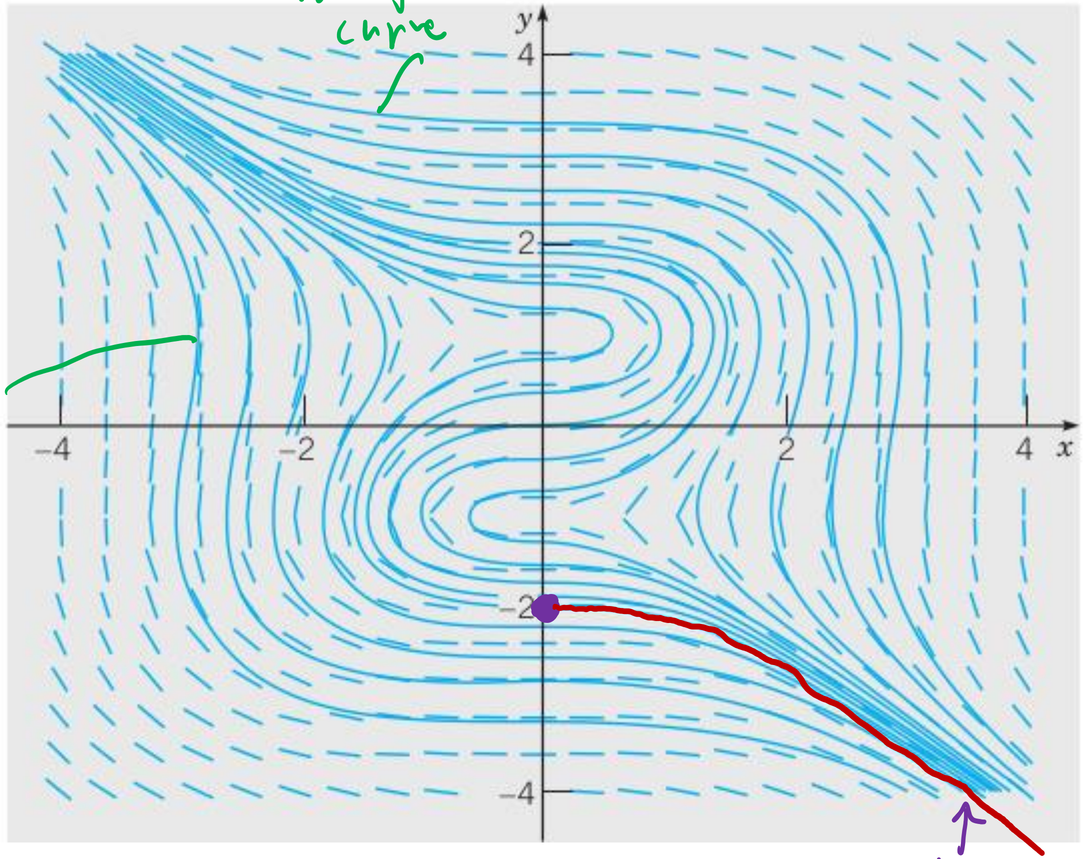
$\frac{1}{3}x^3 - y + \frac{1}{3}y^3 = \tilde{c} \leftarrow G(x,y;\tilde{c}) = 0$

$-x^3 + 3y - y^3 = c, \quad c = -3\tilde{c},$

$$\left\{\begin{array}{l} \frac{dy}{dx} = \frac{x^2}{1-y^2}, \; x \geq 0 \\ y(0) = -2 \end{array}\right\} (*)$$

integral curve

integral curve



solution to $(*)$

SIR Model: Separability

$R_0 = \frac{\beta N}{\gamma}$

Ex: (SIR Model)

$S \xrightarrow{\beta I S} I \xrightarrow{\gamma I} R$

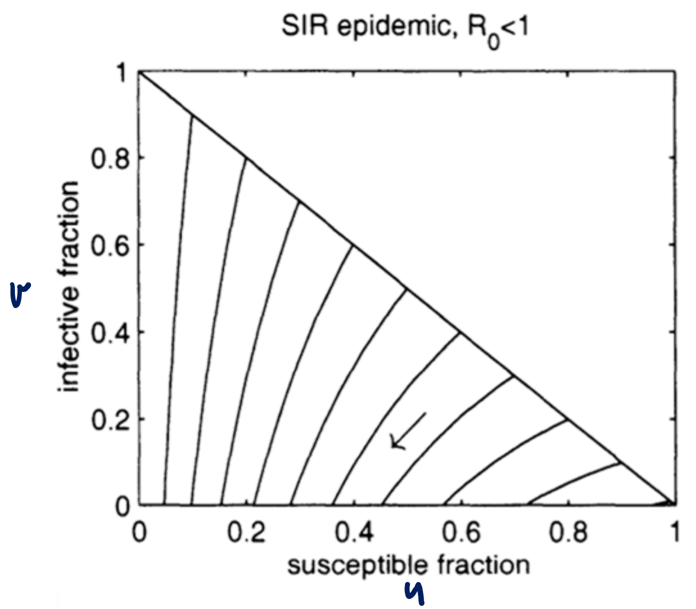
$S(t)$: Susceptible, $I(t)$: Infected, $R(t)$: Recovered

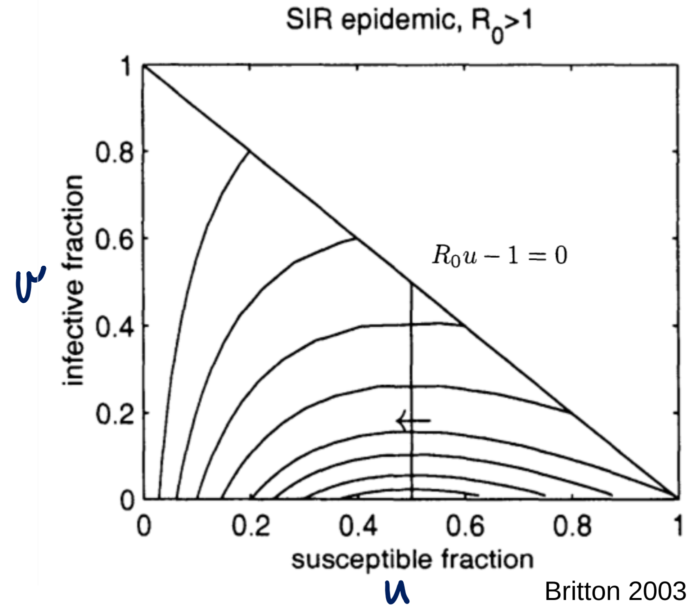
$$\frac{dS}{dt} = -\beta I S, \quad \frac{dI}{dt} = \beta I S - \gamma I, \quad \frac{dR}{dt} = \gamma I$$

Notice: $N(t) = S(t) + I(t) + R(t)$, $\frac{dN}{dt} = 0$, $N(t) = N(0)$, $\forall t$.

$$u = \frac{S}{N}, \quad v = \frac{I}{N}, \quad w = \frac{R}{N}, \quad R_0 = \frac{\beta N}{\gamma}, \quad \tau = \gamma t, \quad u + v + w = 1.$$

$$\frac{du}{d\tau} = -R_0 u v, \quad \frac{dv}{d\tau} = (R_0 u - 1) v, \quad \frac{dw}{d\tau} = v.$$





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SIR Model: Separability

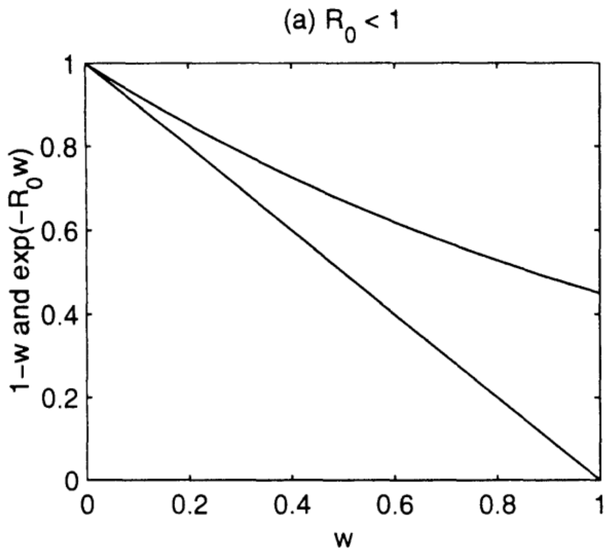
Ex: (SIR Model)

$S \xrightarrow{\beta I S} I \xrightarrow{\gamma I} R$

$S(t)$: Susceptible, $I(t)$: Infected, $R(t)$: Recovered

$R_0 = \frac{\beta N}{\gamma}$, $\tau = \gamma t$, $u + v + w = 1$.

$\frac{du}{d\tau} = -R_0 u v$, $\frac{dv}{d\tau} = (R_0 u - 1) v$, $\frac{dw}{d\tau} = v$.



What are the total number of people infected over the epidemic?

$\frac{\frac{dw}{d\tau}}{\frac{du}{d\tau}} = \frac{dw}{d\tau}\frac{d\tau}{du} = \frac{dw}{du} = -\frac{1}{R_0 u}$, $\frac{\frac{dv}{d\tau}}{\frac{du}{d\tau}} = \frac{dv}{d\tau}\frac{d\tau}{du} = \frac{dv}{du} = -1 + \frac{1}{R_0 u}$, $\left\{ \begin{array}{l} \frac{dw}{du} = -\frac{1}{R_0 u} \\ \frac{dv}{du} = -1 + \frac{1}{R_0 u} \end{array} \right\}$

$\Rightarrow \frac{du}{dw} = -R_0 u \Rightarrow \boxed{u = e^{-R_0 w}}$ $(*)$,

We want to know total # infected

Hence, $w(t) \to w_1$, $t \to \infty$, $v(t) \to 0$, $u(t) \to u_1$. $w_1 \cdot N$ = total # infected

$t \to \infty$, $(u, v, w) = (u_1, 0, w_1)$, $u_1 = 1 - w_1$

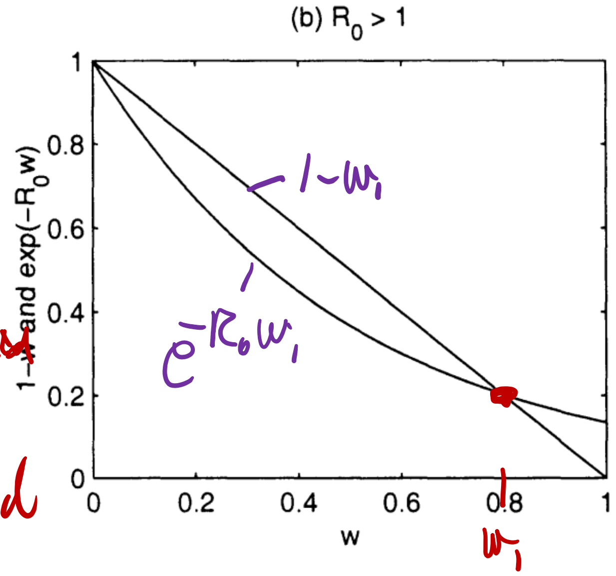
total fraction infected is

$\boxed{1 - w_1 = e^{-R_0 w_1}}$

$u + v + w = 1$

$v = 1 - u - w = 1 - e^{-R_0 w} - w$,

$\boxed{\frac{dw}{d\tau} = 1 - w - e^{-R_0 w}}$



Summary

Integrating Factor
Separation of Variables

Integrator Factor Method

Summary of steps

Can the differential equation be put in the following form?

$$\frac{dy}{dt} + p(t)y = g(t)$$

Yes:

1. Compute the integrating factor

$$\mu(t) = \exp\left(\int p(s)ds\right)$$

2. Integrate the full solution

$$y(t) = \frac{C}{\mu(t)} + \frac{1}{\mu(t)}\int \mu(s)g(s)ds$$

3. Solve for C using initial condition $y(t_0) = y_0$

No: then must use another method.

Ex! $\frac{dy}{dt} = -2y + e^{-t}, \quad y(0) = 3$

yes, $\mu(t) = e^{\int 2\, ds} = e^{2t}$

$$y(t) = \frac{C}{e^{2t}} + \frac{1}{e^{2t}}\int e^{2s}e^{-s}ds$$

$$= \tilde{C}e^{-2t} + e^{-2t}\int e^{s}ds$$

$$= \tilde{C}e^{-2t} + e^{-2t}e^{t}$$

$$= \tilde{C}e^{-2t} + e^{-t}.$$

$$y(t_0) = \tilde{C}e^{-2t_0} + e^{-t_0} = y_0 = 3$$

$$y(0) = \tilde{C}e^{0} + e^{-0} = y_0 = 3$$

$$= \tilde{C}\cdot 1 + 1 = 3 \Rightarrow \tilde{C} = 2$$

$$\boxed{y(t) = 2e^{-2t} + e^{-t}}$$ ✓ ✓

Separation of Variables

Summary of steps:

Can the differential equation be put in the following form?

$$\frac{dy}{dx} = \frac{-M(x)}{N(y)}$$

Yes:

1. Compute the relationship for x,y

$$h(x,y) = \int M(x)dx + \int N(y) = C \longrightarrow h(x,y(x)) = C$$

2. Solve for y(x) from h(x,y) = C.

3. Solve for C using initial condition $y(x_0) = y_0$

No: then must use another method.

Integrating Factor: Example

$$\left\{ \begin{array}{l} \frac{dy}{dx} = -2y + e^{-3x} \\ y(0) = -1 \end{array} \right\}$$

$$\frac{dy}{dt} + p(t)y = g(t)$$

$$\frac{dy}{dt} + 2y = e^{-3t} \Rightarrow p(t) = 2, \quad g(t) = e^{-3t}$$

$$u(t) = e^{\int p(s)ds} = e^{\int 2 ds} = e^{2t}, \quad y(t) = ce^{-2t} - e^{-3t}$$

$$y(t) = \frac{c}{u(t)} + \frac{1}{u(t)} \int u(s) g(s) ds$$

$$= \frac{c}{e^{2t}} + \frac{1}{e^{2t}} \int e^{2s} e^{-3s} ds$$

$$= ce^{-2t} + e^{-2t} \int e^{-s} ds = ce^{-2t} + e^{-2t}[-e^{-t}]$$

$$y(0) = -1$$

$$y(0) = ce^{0} - e^{0}$$

$$= c \cdot 1 - 1 = -1$$

$$= c - 1 = -1 \Rightarrow c = 0$$

$$\boxed{y(t) = -e^{-3t}}$$

$$\frac{dy}{dt} = \frac{d}{dt}(-e^{-3t}) = -(-3)e^{-3t}$$

$$= \underline{3e^{-3t}}$$

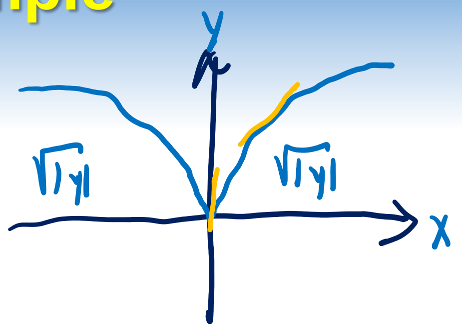
$$-2y + e^{-3t} = -2(-e^{-3t}) + e^{-3t}$$

$$= 2e^{-3t} + e^{-3t} = \underline{3e^{-3t}} \checkmark$$

$$y(0) = -e^{-3 \cdot 0} = -e^{0} = -1 \checkmark$$

Separation of Variables: Example

$$\begin{cases} \dfrac{dy}{dx} = \sqrt{y}, \quad x > 0 \\ y(0) = 0 \end{cases} \quad (*)$$



$\dfrac{dy}{dx} = \dfrac{-M(x)}{N(y)}$, yes $M(x) = -1$, $N(y) = \dfrac{1}{\sqrt{y}}$,

$$\frac{dy}{dx} = \sqrt{y}$$

$$\frac{1}{\sqrt{y}}\,dy = dx$$

$$-1\,dx + \frac{1}{\sqrt{y}}\,dy = 0$$

$$\int -1\,dx + \int \frac{1}{\sqrt{y}}\,dy = c$$

$$h(x,y) = \int -1\,dx + \int \frac{1}{\sqrt{y}}\,dy = c$$

$$= -x + 2\sqrt{y} = c$$

$$\sqrt{y} = \frac{1}{2}x + \frac{1}{2}c = \frac{1}{2}(x+c)$$

$$y(x) = \frac{1}{4}(x+c)^2$$

$y(0) = 0$, $y(0) = \frac{1}{4}(0+c)^2 = \frac{1}{4}c^2 = 0$, $\boxed{y(x) = \frac{1}{4}x^2}$ ✓ ✓

$\Rightarrow c = 0$

$$\frac{dy}{dx} = \frac{1}{4}\frac{d}{dx}x^2 = \frac{1}{4}(2x) = \frac{1}{2}x$$

$$\sqrt{y} = \sqrt{\frac{1}{4}x^2} = \frac{1}{2}|x|.$$

Remark: There are actually many solutions to $(*)$

$\boxed{y(x) \equiv 0}$ ✓ ✓

Solutions to differential equations need not be unique!

$$y(x) = \begin{cases} 0, & x < x_c \\ \frac{1}{4}(x - x_c)^2, & x \geq x_c \end{cases}. \quad x_c \in \mathbb{R}, \; x_c > 0.$$