Differential Equations

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Existence and Uniqueness

Existence and Uniqueness: ODEs

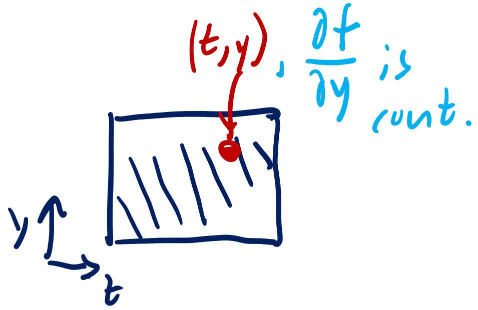
$$y' = f(t, y), \quad y(0) = y_0$$

Def: (Solution to ODE) We say that $y(t)$ is a *solution to the ODE* on an interval I, if

(i) $y(t)$ is differentiable on I

(ii) $y'(t) = f(t, y(t))$

(iii) $y(0) = y_0$.



Def: We say that $f(t,y)$ is *continuously differentiable* on a region R if $\frac{\partial f}{\partial y}$ is continuous for all $y \in R$.

$R = \{(t,y) \in \mathbb{R}^2 \mid t_1 \leq t \leq t_2,\ y_1 \leq y \leq y_2\}$

Theorem: If both f and $\frac{\partial f}{\partial y}$ are continuous on a rectangle $R : |t| \leq a, |y| \leq b$, then there exists a unique solution $y(t)$ of the DE on some interval $|t| \leq h \leq a$.

Remark: Above are sufficient conditions for existence and uniqueness.

If above are not satisfied, then inconclusive. Need additional theory to determine behaviors.

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Sketch of proof:

- $\frac{dy}{dt} = f(t,y)$, by the Fundamental Theorem of Calculus $\Rightarrow$ $y(t) = y(0) + \int_0^t f(s, y(s))\,ds$, $y(0) = y_0$

  $y(t) - y(0) = \int_0^t \frac{dy}{ds}\,ds = \int_0 f(s, y(s))\,ds$

- Let $T\phi = y_0 + \int_0^t f(s, \phi(s))\,ds$. Now solving the ODE corresponds to finding a fixed point of T. $T\phi^* = \phi^*$, $\phi^*(t) = T\phi^* = y_0 + \int_0^t f(s, \phi^*(s))\,ds$, $g(t) = (T\phi)(t)$.

- Try to show that T is a contraction map. $\|T(\phi_1 - \phi_2)\| \leq \lambda \|\phi_1 - \phi_2\|$, $\lambda < 1$.
  $\|g\|^2 = \int g^2(s)\,ds$, Theory that shows contraction maps have unique fixed pts.

- As a conclusion there exists a unique ϕ^*, let $y(t) = \phi^*(t)$. ∎

Remark: The fixed point can often be approximated by $\phi_{k+1} = T\phi_k$, ϕ_0, $\phi^* = \lim_{k\to\infty} \phi_k$.

$\phi_k = T^k \phi_0$, $\phi^* = \lim_{k\to\infty} \phi_{k+1}$
$= y_0 + \int_0^t \lim_{k\to\infty} f(s, \phi_k(s))\,ds$
$= y_0 + \int_0^t f(s, \phi^*(s))\,ds$.

Picard Iteration : Example

Ex: $y' = ay, \quad y(0) = y_0$ $\rightarrow$ $y(t) = y_0 e^{at}, \quad e^{at} = \sum_{\ell=0}^{\infty} \frac{(at)^\ell}{\ell!}.$

Picard Iteration: $y(t) = y(0) + \int_0^t a\, y(s)\, ds$

$\phi_{k+1}(t) = y_0 + \int_0^t a\, \phi_k(s)\, ds =: T\phi_k \leftarrow$ (Picard Iterations)

$\phi_0(t) = y_0, \quad \phi_1(t) = (T\phi_0)(t) = y_0 + \int_0^t a\, \phi_0(s)\, ds = y_0 + \int_0^t a\, y_0\, ds$

$= y_0(1 + at)$

$\phi_2(t) = (T\phi_1)(t) = y_0 + \int_0^t a\, \phi_1(s)\, ds = y_0 + \int_0^t a(y_0(1+as))\, ds$

$= y_0\left(1 + at + \frac{a^2t^2}{2}\right)$

$\phi_3(t) = y_0 + \int_0^t a\, \phi_2(s)\, ds$

$= y_0\left(1 + at + \frac{a^2t^2}{2} + \frac{a^3t^3}{2 \cdot 3}\right)$

...

$\phi_k(t) = y_0\left(1 + at + \frac{a^2t^2}{2} + \frac{a^3t^3}{2\cdot 3} + \cdots + \frac{a^kt^k}{k!}\right) = y_0 \sum_{\ell=0}^{k} \frac{(at)^\ell}{\ell!}$

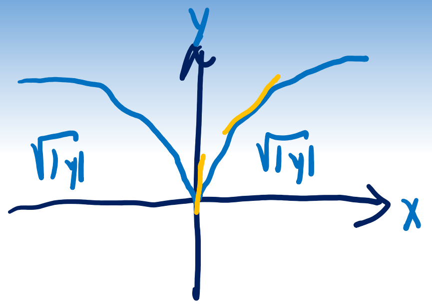
$\phi^* = \lim_{k\to\infty} \phi_k$

$= y_0 \sum_{\ell=0}^{\infty} \frac{(at)^\ell}{\ell!} = y_0 e^{at}$

$\therefore \quad \phi^*(t) = y_0 e^{at}. \quad \checkmark$

Existence and Uniqueness: Examples

Ex: $\left\{\begin{aligned} &\frac{dy}{dx} = \sqrt{y}, \quad x > 0. \\ &y(0) = 0 \end{aligned}\right\} \quad (*)$



Solutions

$\boxed{y(x) = \frac{1}{4}x^2}$

$\boxed{y(x) \equiv 0}$

$$\boxed{y(x) = \begin{cases} 0, & x < x_c \\ \frac{1}{4}(x - x_c)^2, & x \geq x_c \end{cases} \quad \begin{aligned} &x_c \in \mathbb{R} \\ &x_c > 0. \end{aligned}}$$

No unique solution to this ODE at y(0) = 0!

Behavior of f(y)

$\frac{dy}{dx} = f(y), \quad f(y) = \sqrt{y}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y^{1/2}) = \frac{1}{2}y^{-1/2}$

$= \frac{1}{2}\frac{1}{\sqrt{y}}.$

At $x = 0, y = 0$ the function $\frac{\partial f}{\partial y}$ is **not** continuously differentiable in any region $R \ni (0,0)$.

Existence and Uniqueness: Examples

Ex 1: $\left\{ \begin{array}{l} \frac{dy}{dx} = \frac{x^2}{1-y^2} \\ y(0) = -2 \end{array} \right\} (*)$

$R = \{(x,y) \mid -\frac{1}{2} \leq x \leq \frac{1}{2}, \frac{-5}{2} \leq y \leq \frac{-3}{2}\}$

$(x,y) \in R$, $f(x,y) = \frac{x^2}{1-y^2}$,

$\frac{\partial f}{\partial y} = \frac{-x^2}{(1-y^2)^2}(-2y) = \frac{+2x^2y}{(1-y^2)^2}$

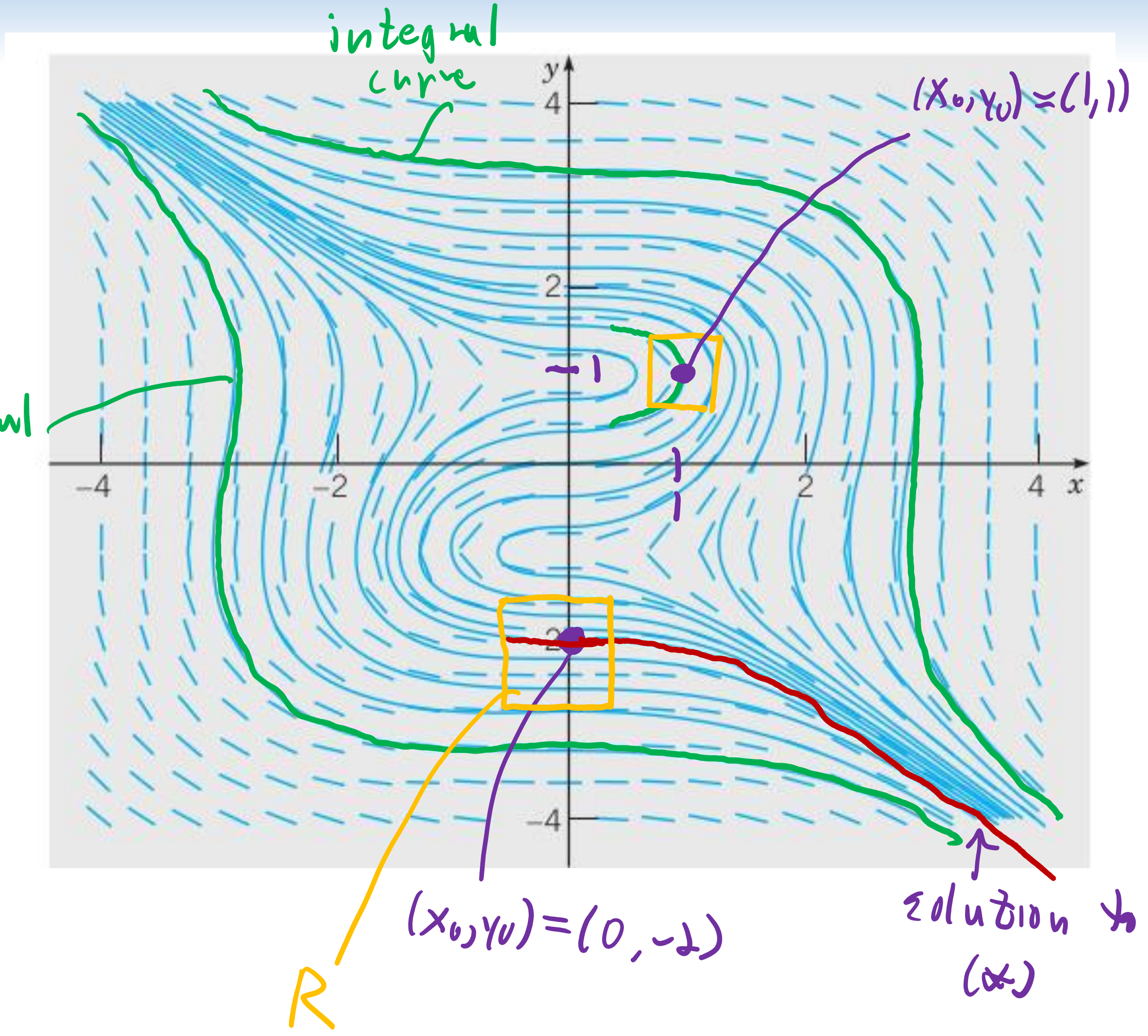
$\Rightarrow$ $y \in R$ $f, \frac{\partial f}{\partial y}$ are continuous.

$\Rightarrow$ **Theorem gives that there exists a unique solution.**

Ex 2: $\left\{ \begin{array}{l} \frac{dy}{dx} = \frac{x^2}{1-y^2} \\ y(1) = 1 \end{array} \right\} (*)$

There is no region R for which $(x_0,y_0) \in R$, $f, \frac{\partial f}{\partial y}$ is continuous,

This means the theorem does not apply to this situation. Inconclusive.



Existence and Uniqueness: ODEs (Summary)

Summary of steps:

Put the differential equation into the following form

$$y' = f(t, y), \quad y(0) = y_0$$

1. Check if f(t,y) is continuous in rectangle R.

2. Check if $\frac{\partial f}{\partial y}$ is continuous in rectangle R.

Do both 1 and 2 hold?

Yes: then theorem gives the **solution exists and is unique.**

No: then **inconclusive** and **must use other theory** to determine behaviors.

Existence and Uniqueness: Examples

Summary of steps:

Put the differential equation into the following form

$$y' = f(t,y), \quad y(0) = y_0$$

1. Check if f(y) is continuous in rectangle R.

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Do both 1 and 2 hold?

Yes: then theorem gives the **solution exists and is unique.**

No: then **inconclusive** and **must use other theory** to determine behaviors.

Ex: $y' = (1+t^2)y^2$, $y(0) = 1$

We need to find R which contains $R \ni (t_0, y_0) = (0,1)$.

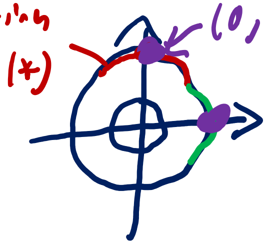
$f(t,y) = (1+t^2)y^2$, $\frac{\partial f}{\partial y} = 2(1+t^2)y$

Let $R = \{(t,y) \mid -1 \le t \le 1,\ 0 \le y \le 2\}$

Ex:

then f, $\frac{\partial f}{\partial y}$ are both continuous in R. Therefore, the thm. holds so there exists a unique solution $y(t)$ for $|t| \le h$, $h > 0$.

Ex: $\left\{ y' = -\frac{x}{y},\ y(0) = 1 \right\}$ (*)



Can we find R s.t. $R \ni (x_0, y_0) = (0,1)$, and f, $\frac{\partial f}{\partial y}$ are continuous?

$f(x,y) = -\frac{x}{y}$, $\frac{\partial f}{\partial y} = +\frac{x}{y^2}$.

Let $R = \{(x,y) \mid -1 \le t \le 1,\ \frac{1}{2} \le y \le \frac{3}{2}\}$

then both f, $\frac{\partial f}{\partial y}$ are continuous in R. Therefore, there exists a unique solution $y(t)$ for $|t| \le h$, $h > 0$.

Ex: $\left\{ y' = -\frac{x}{y},\ y(1) = 0 \right\}$

$\tilde{x} = x - 1$, $\tilde{y} = y$

$\tilde{y}' = y'$, $x = \tilde{x} + 1$

$\tilde{y}' = -\frac{(\tilde{x}+1)}{\tilde{y}}$, $\tilde{y}(0) = 0$.

$(\tilde{x}_0, \tilde{y}_0) = (0,0)$

$\tilde{f} = -\frac{(\tilde{x}+1)}{\tilde{y}}$, $\frac{\partial \tilde{f}}{\partial \tilde{y}} = \frac{(\tilde{x}+1)}{\tilde{y}^2}$

$\tilde{f}(0,\tilde{y}) = \frac{-1}{\tilde{y}}$, $\frac{\partial \tilde{f}(0,\tilde{y})}{\partial \tilde{y}} = -\frac{1}{\tilde{y}^2}$

No suitable R exists!

Numerical Methods for ODEs

Numerical Approximation: Euler's Method

Ordinary Differential Equation

$$\frac{d\mathbf{y}}{dt} = f(\mathbf{y})$$
$$\mathbf{y}(0) = \mathbf{y}_0$$

The derivative is by definition

$$\frac{d\mathbf{y}(t)}{dt} = \lim_{h\to 0} \frac{\mathbf{y}(t+h) - \mathbf{y}(t)}{h}$$

Approximate the solution at discrete times by

$$\frac{d\mathbf{y}(t_n)}{dt} \approx \frac{\mathbf{y}(t_{n+1}) - \mathbf{y}(t_n)}{h} \;, \qquad \frac{d\mathbf{y}(t_n)}{dt} = \mathbf{f}(\mathbf{y}(t_n))$$

$$t_k = kh \;, \quad h = t_{n+1} - t_n$$

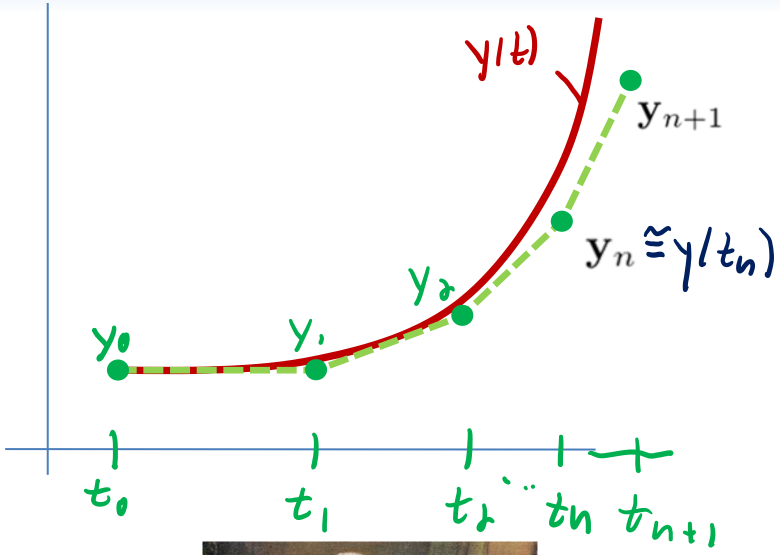
This gives

$$\frac{\mathbf{y}(t_{n+1}) - \mathbf{y}(t_n)}{h} \approx \mathbf{f}(\mathbf{y}(t_n)) \longrightarrow \mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(\mathbf{y}_n)$$

How accurate is this discrete equation? How robust to numerical errors?

More sophisticated approximations also possible (improve accuracy, stability, …)

Finite Difference Approximation



Leonard Euler, 1707-1783

$$\boxed{\begin{array}{c} y' = f(y) \\ \text{Euler's Method} \\ w_{n+1} = w_n + hf(w_n) \end{array}}$$

$$w_n \approx y(t_n)$$

$$\boxed{\begin{array}{c} y' = f(t, y) \\ \text{Euler's Method} \\ w_{n+1} = w_n + hf(t_n, w_n) \end{array}}$$

$$w_n \approx y(t_n)$$