Differential Equations

Paul J. Atzberger
Department of Mathematics
University of California Santa Barbara

Numerical Methods for ODEs

Numerical Approximation: Euler's Method

Ordinary Differential Equation

$$\frac{d\mathbf{y}}{dt} = f(\mathbf{y})$$
$$\mathbf{y}(0) = \mathbf{y}_0$$

The derivative is by definition

$$\frac{d\mathbf{y}(t)}{dt} = \lim_{h\to 0}\frac{\mathbf{y}(t+h) - \mathbf{y}(t)}{h}$$

Approximate the solution at discrete times by

$$\frac{d\mathbf{y}(t_n)}{dt} \approx \frac{\mathbf{y}(t_{n+1}) - \mathbf{y}(t_n)}{h}\ , \qquad \frac{d\mathbf{y}(t_n)}{dt} = \mathbf{f}(\mathbf{y}(t_n))$$

$$t_k = kh\ , \quad h = t_{n+1} - t_n$$

This gives

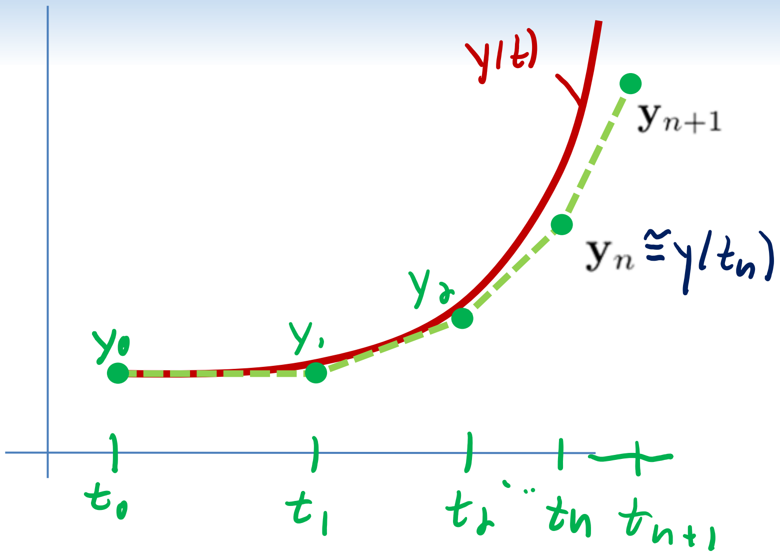
$$\frac{\mathbf{y}(t_{n+1}) - \mathbf{y}(t_n)}{h} \approx \mathbf{f}(\mathbf{y}(t_n))$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(\mathbf{y}_n)$$

How accurate is this discrete equation? How robust to numerical errors?

More sophisticated approximations also possible (improve accuracy, stability, …)

Finite Difference Approximation



Leonard Euler, 1707-1783

$y' = f(y)$	$y' = f(t, y)$
Euler's Method	Euler's Method
$w_{n+1} = w_n + hf(w_n)$	$w_{n+1} = w_n + hf(t_n, w_n)$

$w_n \approx y(t_n)$ $\qquad$ $w_n \approx y(t_n)$

Numerical Approximation: Euler's Method

Ex: $y' = -ay$, $y(0) = y_0$, $a > 0$

solution $y(t) = y_0 e^{-at}$.

$w_{n+1} = w_n - haw_n$, $f(w_n) = -aw_n$

$= (1 - ha) w_n$

$\vdots$

$= (1 - ha)^{n+1} w_0$, $w_0 = y_0$.

Let $T = (n+1)h$ to approximate the solution on $[0, T]$.

$h = \frac{T}{n+1}$, so $w_{n+1} = \left(1 - \frac{aT}{n+1}\right)^{n+1} w_0$, $\lim_{N \to \infty} \left(1 + \frac{s}{N}\right)^N = e^s$

let $h \to 0$ via $h = \frac{T}{n+1}$ as $n \to \infty$, then

$\lim_{h \to 0} w_{n+1} = \lim_{n \to \infty} \left(1 - \frac{aT}{n+1}\right)^{n+1} w_0 = e^{-aT} w_0$. (This shows that Euler's method converges)

How large can h be and still give stable behavior? $|1 - ha| \leq 1$, $-1 \leq 1 - ha \leq 1$

$\lim_{n \to \infty} w_{n+1} = 0$, how large can h be? $w_{n+1} = (1 - ha)^{n+1} w_0$, $-2 \leq -ha \leq 0$, $0 \leq h \leq \frac{2}{a}$. (stability)

Leonard Euler, 1707-1783

$y' = f(y)$
Euler's Method
$w_{n+1} = w_n + hf(w_n)$

$w_n \approx y(t_n)$

$y' = f(t, y)$
Euler's Method
$w_{n+1} = w_n + hf(t_n, w_n)$

$w_n \approx y(t_n)$

Numerical Packages: Solve Differential Equations in Python

Ex: $y'(t) = -ay,\ y(0) = 3.$

python SciPy

Python code for approximating solution.

```
# Codes for integrating and plotting ODEs using python
#
# Paul J. Atzberger
#

import numpy as np;
import matplotlib;
import matplotlib.pyplot as plt;
from scipy.integrate import odeint
```

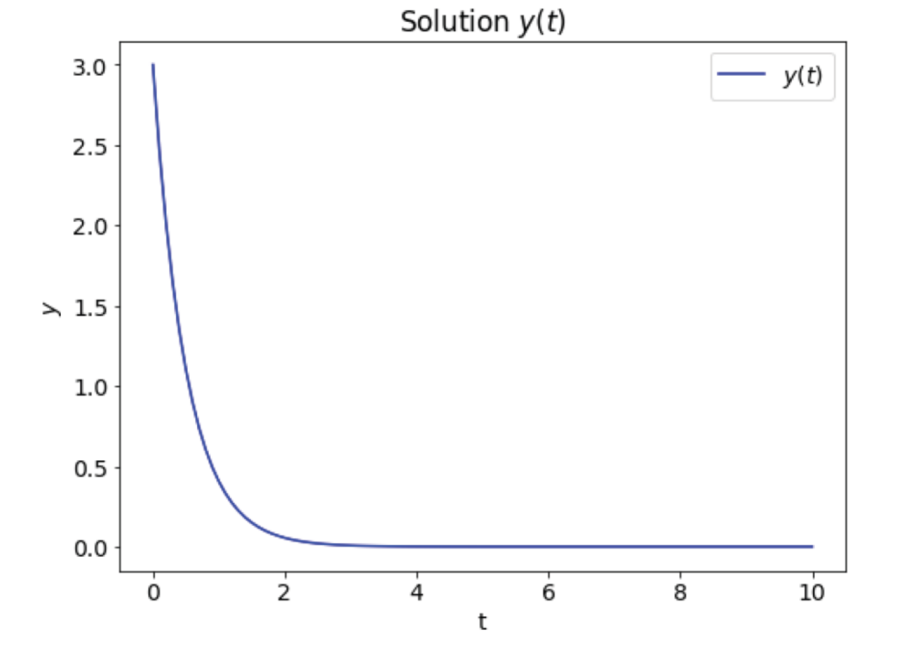
```
fontsize = 14;
font = {'family' : 'DejaVu Sans',
        'weight' : 'normal',
        'size'   : fontsize};

matplotlib.rc('font', **font);
```

```
# setup the function f(y,t) for RHS dy/dt = f(y,t).
def f1(y, t, a):
    f = -a*y;
    return f;
```

```
# setup the initial value problem
a = 2.0;
y0 = 3.0;
```

```
# construct approximation to the solution of the ODE
t = np.linspace(0, 10, 101);
sol = odeint(f1, y0, t, args=(a,));
```

```
# plot the solution
plt.figure(1,figsize=(8,6),facecolor='white');
plt.plot(t,sol[:,0],'b',label=r'$y(t)$');
plt.legend(loc='best');
plt.xlabel('t')
plt.ylabel(r'$y$')
plt.title(r'Solution $y(t)$');
#plt.grid();
plt.draw();
```



Second Order Differential Equations

Second Order Differential Equations

Def: A *Second Order Differential Equation* refers to equations that can be put into the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right).$$

Def: A Second Order Differential Equation is called *linear* if it can be put into the form

$$\frac{d^2y}{dt^2} = \alpha_1(t) + \alpha_2(t)y + \alpha_3(t)\frac{dy}{dt}.$$

Def: Second Order Differential Equations that **can not be put into the above form** are called *non-linear*.

Remark: For convenience we will often express linear equations as $f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y$, giving $y'' + p(t)y' + q(t)y = g(t)$.

Def: The *Initial Value Problem (IVP)* for Second Order Differential Equations are problems of the form

$$\begin{cases} \frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right), \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases}$$

Second Order Differential Equations

Consider a *linear* second order differential equation

$$\frac{d^2y}{dt^2} = \alpha_1(t) + \alpha_2(t)y + \alpha_3(t)\frac{dy}{dt}.$$

Def: Second Order Differential Equations are called *homogeneous* when $\alpha_1(t) = 0$.

Remark: When expressing linear equations as $y'' + p(t)y' + q(t)y = g(t)$, it is **homogeneous** when $g(t) = 0$, giving $y'' + p(t)y' + q(t)y = 0$.

Def: Second Order Differential Equations with $\alpha_1(t) \neq 0$ are called *inhomogeneous* or *nonhomogeneous*.

Second Order Differential Equations

Ex! $y'' = y, \quad y(0) = -3, \quad y'(0) = 1$

$y'' - y = 0, \quad \frac{d^2 y}{dt^2} - y = 0, \quad \left(\frac{d^2}{dt^2} - 1\right) y = 0$

$\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} + 1\right) y = 0 \rightarrow \left(\frac{d}{dt} + 1\right) y = 0 \rightarrow \frac{dy}{dt} = -y \rightarrow y_2(t) = c_2 e^{-t}$

$\left(\frac{d}{dt} + 1\right)\left(\frac{d}{dt} - 1\right) y = 0 \rightarrow \left(\frac{d}{dt} - 1\right) y = 0 \rightarrow \frac{dy}{dt} = y \rightarrow y_1(t) = c_1 e^{t}$

$\bar{y}(t) = c_1 e^{t} + c_2 e^{-t}, \quad \bar{y}'' - \bar{y} = 0$ ✓

$\bar{y}(0) = c_1 e^{0} + c_2 e^{-0} = c_1 + c_2 = -3. \quad 2c_1 = -3 + 1 = -2 \Rightarrow c_1 = -1$

$\bar{y}'(0) = c_1 e^{0} - c_2 e^{-0} = c_1 - c_2 = 1, \quad 2c_2 = -3 - 1 = -4 \Rightarrow c_2 = -2$

$\bar{y}(t) = -e^{t} - 2e^{-t}$ ✓

Second Order Differential Equations

Consider Linear Homogeneous Second order Differential Equation with constant coefficients.

$$ay'' + by' + cy = 0$$

We can try to solve using a similar strategy as our example.

$$\left(a\frac{d^2}{dt^2} + b\frac{d}{dt} + c\right)y = 0, \quad \text{formally } s = \frac{d}{dt}, \quad (as^2 + bs + c) = (a(s-r_1)(s-r_2))$$

$$ar^2 + br + c = 0, \quad a(r-r_1)(r-r_2) = ar^2 + br + c$$

↳ characteristic equation.

$$a\left(\frac{d}{dt} - r_1\right)\left(\frac{d}{dt} - r_2\right)y = 0 \to \left(\frac{d}{dt} - r_2\right)y = 0 \to \frac{dy}{dt} = r_2 y \to y' = r_2 y \to y_2(t) = c_2 e^{r_2 t}$$

$$a\left(\frac{d}{dt} - r_2\right)\left(\frac{d}{dt} - r_1\right)y = 0 \to \left(\frac{d}{dt} - r_1\right)y = 0 \to \frac{dy}{dt} = r_1 y \to y' = r_1 y \to y_1(t) = c_1 e^{r_1 t}$$

Candidate solution

$$\bar{y}(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Second Order Differential Equations

Ex: $y'' + 3y' + 2y = 0, \quad y(0) = 3, \quad y'(0) = -5$

$ay'' + by' + c = 0$

$ar^2 + br + c = 0$

$a = 1, \quad b = 3, \quad c = 2$

$y(t) = \tilde{c}e^{rt}, \quad y' = r\tilde{c}e^{rt}, \quad y'' = r^2\tilde{c}e^{rt}$

$r^2(\tilde{c}e^{rt}) + 3r(\tilde{c}e^{rt}) + 2(\tilde{c}e^{rt}) = 0$

$r^2 + 3r + 2 = 0$

$$r = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm \sqrt{1}}{2} = \begin{cases} r_1 = -1 \\ r_2 = -2 \end{cases}$$

$\bar{y}(t) = c_1 e^{-t} + c_2 e^{-2t}$

$\bar{y}(0) = c_1 + c_2 = 3$

$\bar{y}'(0) = -c_1 - 2c_2 = -5, \quad -c_2 = -5 + 3 = -2 \Rightarrow c_2 = 2$

$c_1 = 3 - c_2 = 3 - 2 = 1 \Rightarrow c_1 = 1$

$\bar{y}(t) = e^{-t} + 2e^{-2t}$

Summary of steps:

Can the differential equation be put into the following form?

$$ay'' + by' + cy = 0$$

Yes: Consider the characteristic equation and solve for r

$$ar^2 + br + c = 0$$

Are the roots r_1 and r_2 real-valued and distinct?

Yes: then the **solution is of the following form**

$$y(t) = c_1\exp(r_1 t) + c_2\exp(r_2 t).$$

Solve the IVP by using $y(t_0) = y_0, y'(t_0) = y_0'$ and solving for c_1, c_2.

No: then **must use further theory** to determine solution (developed in later lectures).