Differential Equations

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Second Order Differential Equations

Second Order Differential Equations

Def: A *Second Order Differential Equation* refers to equations that can be put into the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right).$$

Def: A Second Order Differential Equation is called *linear* if it can be put into the form

$$\frac{d^2y}{dt^2} = \alpha_1(t) + \alpha_2(t)y + \alpha_3(t)\frac{dy}{dt}.$$

Def: Second Order Differential Equations that **can not be put into the above form** are called *non-linear*.

Remark: For convenience we will often express linear equations as $f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y$, giving $y'' + p(t)y' + q(t)y = g(t)$.

Def: The *Initial Value Problem (IVP)* for Second Order Differential Equations are problems of the form

$$\begin{cases} \frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right), \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases}$$

Second Order Differential Equations

Consider a *linear* second order differential equation

$$\frac{d^2y}{dt^2} = \alpha_1(t) + \alpha_2(t)y + \alpha_3(t)\frac{dy}{dt}.$$

Def: Second Order Differential Equations are called *homogeneous* when $\alpha_1(t) = 0$.

Remark: When expressing linear equations as $y'' + p(t)y' + q(t)y = g(t)$, it is **homogeneous** when $g(t) = 0$, giving $y'' + p(t)y' + q(t)y = 0$.

Def: Second Order Differential Equations with $\alpha_1(t) \neq 0$ are called *inhomogeneous* or *nonhomogeneous*.

Second Order Differential Equations

Real-Valued Distinct Roots

Second Order Differential Equations

Ex! $y'' = y, \quad y(0) = -3, \quad y'(0) = 1$

$$y'' - y = 0, \quad \frac{d^2 y}{dt^2} - y = 0, \quad \left(\frac{d^2}{dt^2} - 1\right) y = 0$$

$$\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} + 1\right) y = 0 \rightarrow \left(\frac{d}{dt} + 1\right) y = 0 \rightarrow \frac{dy}{dt} = -y \rightarrow y_2(t) = c_2 e^{-t}$$

$$\left(\frac{d}{dt} + 1\right)\left(\frac{d}{dt} - 1\right) y = 0 \rightarrow \left(\frac{d}{dt} - 1\right) y = 0 \rightarrow \frac{dy}{dt} = y \rightarrow y_1(t) = c_1 e^{t}$$

$$\bar{y}(t) = c_1 e^{t} + c_2 e^{-t}, \quad \bar{y}'' - \bar{y} = 0 \checkmark$$

$$\bar{y}(0) = c_1 e^{0} + c_2 e^{-0} = c_1 + c_2 = -3. \qquad 2c_1 = -3 + 1 = -2 \Rightarrow c_1 = -1$$

$$\bar{y}'(0) = c_1 e^{0} - c_2 e^{-0} = c_1 - c_2 = 1, \qquad 2c_2 = -3 - 1 = -4 \Rightarrow c_2 = -2$$

$$\bar{y}(t) = -e^{t} - 2e^{-t} \checkmark$$

Second Order Differential Equations

Consider Linear Homogeneous Second order Differential Equations with constant coefficients.

$$ay'' + by' + cy = 0$$

We can try to solve using a similar strategy as our example.

$\underline{\left(a\frac{d^2}{dt^2} + b\frac{d}{dt} + c\right)y} = 0$, formally $s = \frac{d}{dt}$, $(as^2 + bs + c) = (a(s - r_1)(s - r_2))$

$\underline{ar^2 + br + c = 0}$, $a(r - r_1)(r - r_2) = ar^2 + br + c$

↳ characteristic equation.

$$\underline{a\left(\frac{d}{dt} - r_1\right)\left(\frac{d}{dt} - r_2\right)y} = 0 \rightarrow \left(\frac{d}{dt} - r_2\right)y = 0 \rightarrow \frac{dy}{dt} = r_2 y \rightarrow y' = r_2 y \rightarrow y_2(t) = c_2 e^{r_2 t}$$

$$\underline{a\left(\frac{d}{dt} - r_2\right)\left(\frac{d}{dt} - r_1\right)y} = 0 \rightarrow \left(\frac{d}{dt} - r_1\right)y = 0 \rightarrow \frac{dy}{dt} = r_1 y \rightarrow y' = r_1 y \rightarrow y_1(t) = c_1 e^{r_1 t}$$

Candidate solution

$$\bar{y}(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Second Order Differential Equations

Ex: $y''+3y'+2y=0,\ y(0)=3,\ y'(0)=-5$

$y(t)=\tilde{c}e^{rt},\quad y'=r\tilde{c}e^{rt},\quad y''=r^2\tilde{c}e^{rt}$

$r^2(\tilde{c}e^{rt})+3r(\tilde{c}e^{rt})+2(\tilde{c}e^{rt})=0$

$\underline{r^2+3r+2=0}$

$$r=\frac{-3\pm\sqrt{9-8}}{2}=\frac{-3\pm\sqrt{1}}{2}=\begin{cases} r_1=-1 \\ r_2=-2 \end{cases}$$

$\bar{y}(t)=c_1e^{-t}+c_2e^{-2t}$

$\bar{y}(0)=c_1+c_2=3$

$\bar{y}'(0)=-c_1-2c_2=-5,\quad -c_2=-5+3=-2 \Rightarrow c_2=2$

$c_1=3-c_2=3-2=1 \Rightarrow c_1=1$

$\bar{y}(t)=e^{-t}+2e^{-2t}$

$ay''+by'+c=0$

$ar^2+br+c=0$

$a=1,\ b=3,\ c=2$

Second Order Differential Equations

Real-Valued Distinct Roots

Summary of steps:

Can the differential equation be put into the following form?

$$ay'' + by' + cy = 0$$

Yes: Consider the characteristic equation and solve for r

$$ar^2 + br + c = 0$$

Are the roots r_1 and r_2 real-valued and distinct?

Yes: then the **solution is of the following form**

$$y(t) = c_1 \exp(r_1 t) + c_2 \exp(r_2 t).$$

Solve the IVP by using $\mathrm{y}(\mathrm{t}_0) = \mathrm{y}_0, y'(t_0) = y_0'$ and solving for c_1, c_2.

No: then **must use further theory** to determine solution (developed in later lectures).

Second Order Differential Equations

Complex-Valued Distinct Roots

Second Order Differential Equations

Euler Identity

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \theta^{2\ell}}{(2\ell)!} + i \sum_{\ell=1}^{\infty} \frac{(-1)^\ell \theta^{2\ell-1}}{(2\ell-1)!} = \cos(\theta) + i\sin\theta$$

$i = \sqrt{-1}$

$$\cos(\theta) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \theta^{2\ell}}{(2\ell)!}, \quad \sin(\theta) = \sum_{\ell=1}^{\infty} \frac{(-1)^\ell \theta^{2\ell-1}}{(2\ell-1)!}$$

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$

Remark: $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$

$$e^{a+ib} = e^a e^{ib} = e^a(\cos(b) + i\sin(b))$$

$\frac{d}{dt} e^{rt} = r e^{rt}$, when r is complex.

Second Order Differential Equations

Ex: $\left\{ \begin{array}{l} y'' = -2y' - 2y \\ y(0) = 3, \ y'(0) = 1 \end{array} \right\}$

$$ay'' + by' + c = 0$$

$$y'' + 2y' + 2y = 0$$

$$r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 2}}{2}$$

$$= -1 \pm \sqrt{-1}$$

$$= -1 \pm i$$

$$r_1 = -1 + i, \quad r_2 = -1 - i$$

$$\bar{y}(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$\bar{y}(t) = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t}$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\bar{y}(t) = c_1 e^{-t}(\cos(\theta_1) + i\sin(\theta_1)), \quad \theta_1 = t, \ \theta_2 = -t$$

$$+ c_2 e^{-t}(\cos(\theta_2) + i\sin(\theta_2))$$

$$= \tilde{c}_1 e^{-t}\cos(t) + \tilde{c}_2 e^{-t}\sin(t).$$

$$\tilde{c}_1 = c_1 + c_2$$

$$\tilde{c}_2 = i(c_1 - c_2)$$

$$\bar{y}(t) = \tilde{c}_1 e^{-t}\cos(t) + \tilde{c}_2 e^{-t}\sin(t)$$

$$\bar{y}(0) = \tilde{c}_1 e^{-0} \cdot 1 + 0 = 3$$

$$\bar{y}'(0) = -\tilde{c}_1 e^{-0}\cos(0) - \tilde{c}_1 e^{-0}\sin(0)$$

$$- \tilde{c}_2 e^{-0}\sin(0) + \tilde{c}_2 e^{-0}\cos(0)$$

$$= -\tilde{c}_1 + \tilde{c}_2 = 1$$

$$\tilde{c}_1 = 3$$

$$-\tilde{c}_1 + \tilde{c}_2 = 1 \Rightarrow \tilde{c}_2 = 1 + 3 = 4$$

$$y(t) = 3e^{-t}\cos(t) + 4e^{-t}\sin(t).$$

Second Order Differential Equations

Complex-Valued Distinct Roots

Summary of steps:

Can the differential equation be put into the following form?

$$ay'' + by' + cy = 0$$

Yes: Consider the characteristic equation and solve for r

$$ar^2 + br + c = 0$$

Are the roots r_1 and r_2 complex-valued and distinct? $(\boldsymbol{r_1 = \lambda + i\mu,\ \ r_2 = \lambda - i\mu})$

Yes: then the **solution is of the following form,**

$$y(t) = c_1 \exp(\lambda t)\cos(\mu t) + c_2 \exp(\lambda t)\sin(\mu t).$$

Solve the IVP by using $\mathrm{y}(\mathrm{t}_0) = \mathrm{y}_0, y'(t_0) = y_0'$ and solving for c_1, c_2.

No: then **must use further theory** to determine solution.

Second Order Differential Equations

Repeated Roots

Second Order Differential Equations

Ex: $y'' - 2y' + y = 0$

$y(0) = 1, \quad y'(0) = 2$

$y_1(t) = \tilde{c} e^{rt}$

$ar^2 + br + c = 0, \quad a = 1, b = -2, c = 1$

$r^2 - 2r + 1 = 0, \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$r = \frac{+2 \pm \sqrt{4-4}}{2} = 1 \pm 0 \Rightarrow r_1 = 1, r_2 = 1$

naive approach

$\bar{y}(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^t + c_2 e^t = (c_1 + c_2) e^t$

$\bar{y}(0) = c_1 + c_2 = 1, \quad \bar{y}'(0) = c_1 + c_2 = 2$

$\#$ contradiction, no c_1, c_2 exists!

proper approach

$y_1(t) = c_1 e^{r_1 t} = c_1 e^t, \quad y_2(t) = q(t) y_1(t)$

$y_2(t) = q(t) y_1(t)$

$y_2'(t) = q'(t) y_1(t) + q(t) y_1'(t)$

$$y_2''(t) = q''(t) y_1(t) + q'(t) y_1'(t) + q'(t) y_1'(t) + q(t) y_1''(t)$$

$y_1(t) = c_1 e^t$

$y_1'(t) = c_1 e^t.$

$y_2'' - 2y_2' + y_2$

$$= q'' y_1 + q'(\overset{\to 0}{\cancel{2y_1' - 2y_1}}) + q(\overset{\to 0}{\cancel{y_1'' - 2y_1' + y_1}}) = 0$$

$\Rightarrow q'' y_1 = 0 \overset{c_1 \neq 0}{\Rightarrow} q''(t) = 0$

$\Rightarrow q(t) = k_1 + k_2 t, \quad k_1 = 0, k_2 = \tilde{c}_2$

$\Rightarrow y_2(t) = q(t) y_1(t) = \tilde{c}_2 t y_1(t) = c_2 t e^t$

$\bar{y}(t) = c_1 e^t + c_2 t e^t$

solve for initial conditions

$\bar{y}(0) = 1 \Rightarrow c_1 e^0 + c_2 \cdot 0 e^0 = c_1 = 1 \Rightarrow c_1 = 1$

$\bar{y}'(0) = 2 \Rightarrow c_1 e^0 + c_2 e^0 + c_2 \cdot 0 e^0 = 2 \Rightarrow c_2 = 2$

$\bar{y}(t) = e^t + 2te^t.$

Second Order Differential Equations

General Approach

$ay'' + by' + cy = 0, \quad y(0) = y_0, \quad y'(0) = y_0'$

$ar^2 + br + c = 0, \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

repeated when $b^2 - 4ac = 0$

$r_1 = r_2 = \frac{-b}{2a}. \quad y_1(t) = c_1 e^{r_1 t}$

second solution $y_2(t) = q(t)y_1(t)$

$y_2' = q'y_1 + qy_1'$

$y_2'' = q''y_1 + q'y_1' + q'y_1' + qy_1''$

$y_1(t) = c_1 e^{\frac{-b}{2a}t}$

$y_1'(t) = \left(-\frac{b}{2a}\right)c_1 e^{\frac{-b}{2a}t}$

$= \left(-\frac{b}{2a}\right)y_1(t).$

$ay_2'' + by_2' + cy_2 = q''(ay_1) + q'(\cancelto{0}{\underbrace{2ay_1' + by_1}_{2a\left(\frac{-b}{2a}\right)y_1 + by_1}}) + q''(\cancelto{0}{ay_1'' + by_1' + cy_1}) = 0$

$\overset{a \neq 0,\ c_1 \neq 0}{\Longrightarrow} q''(t) = 0 \Rightarrow q(t) = k_1 + k_2 t$

$y_2(t) = c_2 t e^{r_1 t}$

$\bar{y}(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$

initial value problem

$\bar{y}(0) = c_1 e^0 + c_2 \cdot 0 e^0 = y_0$

$\bar{y}'(0) = c_1 r_1 e^0 + c_2 e^0 + c_2 \cdot 0 r_1 e^0 = y_0'$

$\Rightarrow c_1 = y_0, \quad c_2 = y_0'.$

Second Order Differential Equations

Repeated Roots

Summary of steps:

Can the differential equation be put into the following form?

$$ay'' + by' + cy = 0$$

Yes: Consider the characteristic equation and solve for r

$$ar^2 + br + c = 0$$

Are the roots $r_1$and r_2 repeated? $(\boldsymbol{r_1 = r_2 = -b/2a =: r_*})$

Yes: then the **solution is of the following form,**

$$y(t) = c_1 \exp(r_* t) + c_2 t \cdot \exp(r_* t).$$

Solve the IVP by using $\mathrm{y}(\mathrm{t}_0) = \mathrm{y}_0, y'(t_0) = y_0'$ and solving for c_1, c_2.

No: then **must use further theory** to determine solution.

Second Order Differential Equations

Summary of Solution Cases

Second Order Differential Equations

Summary of cases

Steps:

Can the differential equation be put into the following form?

$$ay'' + by' + cy = 0$$

Yes: Consider the characteristic equation and solve for r

$$ar^2 + br + c = 0$$

Cases:

Are the roots	Roots	Solution
real-valued and distinct?	$\boldsymbol{r_1 \neq r_2}$	$y(t) = c_1 \exp(r_1 t) + c_2 \exp(r_2 t)$
complex-valued and distinct?	$\boldsymbol{r_1 = \lambda + i\mu, r_2 = \lambda - i\mu, \mu \neq 0}$	$y(t) = c_1 \exp(\lambda t)\cos(\mu t) + c_2 \exp(\lambda t)\sin(\mu t)$
repeated?	$\boldsymbol{r_1 = r_2 = -b/2a =: r_*}$	$y(t) = c_1 \exp(r_* t) + c_2 t \cdot \exp(r_* t)$

Solve the IVP by using $\mathrm{y}(\mathrm{t}_0) = \mathrm{y}_0, y'(t_0) = y_0'$ and solving for c_1, c_2.

Discriminant	Case
$d = b^2 - 4ac$	discriminant
$d > 0$	real-valued distinct roots
$d < 0$	complex-valued distinct roots
$d = 0$	repeated roots

No: then **must use other theory** to determine solutions.