Differential Equations

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Second Order Differential Equations
Non-Homogeneous Case

Method of Undetermined Coefficients

Method of Undetermined Coefficients

Consider a **non-homogeneous** differential equation

$$y'' + p(t)y' + q(t)y = g(t).$$

Let $L[y] = y'' + p(t)y' + q(t)y = g(t)$, then we can express the differential equation as $L[y] = g(t)$.

Since L is linear, if $Y_1(t)$ and $Y_2(t)$ are solutions then $L[Y_1 - Y_2] = g(t) - g(t) = 0$, so $\psi(t) = Y_1 - Y_2$ is a solution to the homogeneous equation $L[\psi] = 0$.

Theorem If $Y_1(t)$ and $Y_2(t)$ are solutions to the non-homogeneous problem then their difference can be expressed as

$$Y_1(t) - Y_2(t) = \psi(t) = c_1 y_1(t) + c_2 y_2(t),$$

for some choice of c_1, c_2, where $\{y_1(t), y_2(t)\}$ is a fundamental solution set for the homogeneous problem.

Method of Undetermined Coefficients

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Theorem The general solution for $L[y] = g(t)$ to the non-homogeneous problem can be expressed as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + S(t),$$

where $S(t)$ is any specific solution to the non-homogeneous problem $L[S] = g(t)$.

This follows since $L[y] = c_1 L[y_1] + c_2 L[y_2] + L[S]$, $\quad L[y_1] = 0 = L[y_2]$

$= L[S] = g(t) \Rightarrow L[y] = g(t)$ ✓

Method of Undetermined Coefficients

Ex: $\begin{cases} y'' = -2y' - y + e^{-3t} \\ y(0) = 2, \quad y'(0) = 0 \end{cases}$

$y'' + p(t)\,y' + q(t)\,y = g(t)$

$y'' + 2y' + y = e^{-3t}$

Need to find a particular solution $s(t)$

Let $s(t) = Ae^{-3t}$, $L[y] = y'' + 2y' + y$, $L[s] = e^{-3t}$

$L[s] = (9A + 2(-3)A + A)e^{-3t} = e^{-3t}$

$\Rightarrow 9A - 6A + A = 1 \Rightarrow 4A = 1 \Rightarrow A = \frac{1}{4} \Rightarrow$ $\boxed{s(t) = \frac{1}{4}e^{-3t}}$

Now find y_1, y_2 to the homogeneous problem $y'' + 2y' + y = 0$, $L[y] = 0$.

$ar^2 + br + c = 0$, $r^2 + 2r + 1 = 0$, $r = \frac{-2 \pm \sqrt{4-4}}{2} = -1$

(repeated roots) $\rightarrow y_1(t) = e^{-t}$, $y_2(t) = te^{-t}$.

General Solution

$\bar{y}(t) = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{4}e^{-3t}$

Initial Value Problem

$y(0) = 2$, $y'(0) = 0$

$\bar{y}(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 + \frac{1}{4}e^0$

$= c_1 + \frac{1}{4} = 2 \Rightarrow \frac{8}{4} \Rightarrow c_1 = \frac{7}{4}$

$\bar{y}'(0) = -c_1 e^0 + c_2 e^0 - c_2 \cdot 0 e^0$

$+ \frac{-3}{4}e^0 = -c_1 + c_2 - \frac{3}{4} = 0$

$\Rightarrow c_2 = \frac{10}{4} = \frac{5}{2}$

Solution

$\bar{y}(t) = \frac{7}{4}e^{-t} + \frac{5}{2}te^{-t} + \frac{1}{4}e^{-3t}$.

Method of Undetermined Coefficients

Table of a few special solutions to $y'' + p(t)y' + q(t)y = g(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$	$t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases}$	$t^s[(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)e^{\alpha t}\cos\beta t + (B_0 t^n + B_1 t^{n-1} + \cdots + B_n)e^{\alpha t}\sin\beta t]$

Boyce & DePrima 1999

Ex: $y'' - y' + y = e^t\sin(t)$, $S(t) = ?$

$S(t) = Ae^t\sin(t) + Be^t\cos(t)$, $g(t) = e^t\sin(t)$

$S'(t) = Ae^t\sin(t) + Ae^t\cos(t) + Be^t\cos(t) - Be^t\sin(t)$
$= (A-B)e^t\sin(t) + (A+B)e^t\cos(t)$

$S''(t) = (A-B)e^t\sin(t) + (A-B)e^t\cos(t) + (A+B)e^t\cos(t) - (A+B)e^t\sin(t)$
$= -2Be^t\sin(t) + 2Ae^t\cos(t)$

$L[S] = S'' - S' + S$
$= (-2B - (A-B) + A)e^t\sin(t) + (2A - (A+B) + B)e^t\cos(t)$
$= -Be^t\sin(t) + Ae^t\cos(t)$
$= g(t) = e^t\sin(t)$

$\Rightarrow -B = 1$, $A = 0 \Rightarrow B = -1$, $A = 0$

$\Rightarrow S(t) = -e^t\cos(t)$

Second Order Differential Equations
Homogeneous Case

Method of Reduction of Order

Reduction of Order (Derivation)

Homogeneous Differential Equation

$$y'' + p(t)y' + q(t)y = 0 \longrightarrow$$

Suppose One Solution Known

$$y_1(t)$$

Postulated form for second solution

$$y = v(t)y_1(t)$$

Derivatives

$$y' = v'(t)y_1(t) + v(t)y_1'(t)$$

$$y'' = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t).$$

Substituting into differential equation gives

$$y_1 v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0$$

$$y_1 v'' + (2y_1' + py_1)v' = 0$$

Let $w(t) = v'(t)$

First-Order Equation (reduced order)

$$y_1 w' + (2y_1' + py_1)w = 0 \quad \xrightarrow{\text{solve w}} \quad v(t) = \int w(t)dt + C$$

Ex: $2t^2 y'' - t y' - 2y = 0$

$y_1(t) = t^2, \quad y_1' = 2t, \quad y_1'' = 2$

$2t^2 y_1'' - t y_1' - 2y_1 = 2t^2 \cdot 2 - 2t^2 - 2t^2$

$= 4t^2 - 4t^2 = 0$ ✓

$y = v(t) y_1(t)$

$y_1 w' + (2y_1' + p y_1) w = 0,$

$y'' + p(t) y' + q(t)$

$y'' - \frac{1}{2} t^{-1} y' - t^{-2} y = 0, \quad p(t) = -\frac{1}{2} t^{-1}$

$t^2 w' + (4t - \frac{1}{2} t) w = 0$

$w' - \frac{7}{2} t^{-1} w = 0$

$w(t) = e^{\int + \frac{7}{2} t^{-1} dt} = e^{\frac{7}{2} \ln(t)} = t^{7/2}$

$v(t) = \int w(t) dt = \frac{2}{9} t^{9/2}$

$\tilde{y}_2(t) = \frac{2}{9} t^{9/2} t^{4/2} = \frac{2}{9} t^{13/2}, \quad y_2(t) = t^{13/2}.$

Second Order Differential Equations
Non-Homogeneous Case

Method of Variation of Parameters

Variation of Parameters (Derivation)

Non-homogeneous Differential Equation

$$y'' + p(t)y' + q(t)y = g(t)$$

Homogeneous Differential Equation

$$y'' + p(t)y' + q(t)y = 0 \longrightarrow y_c(t) = c_1 y_1(t) + c_2 y_2(t)$$

Postulated form for specific solution

$$y = u_1(t)y_1(t) + u_2(t)y_2(t)$$

Derivatives

$$y' = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t) \longrightarrow y' = u_1(t)y_1'(t) + u_2(t)y_2'(t)$$

letting,

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$

$$y'' = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)$$

Substituting into differential equation gives

$$u_1(t)[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)] + u_2(t)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)] + u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

Conditions on u1 and u2:

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$
$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$

solving $\longrightarrow$

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1,y_2)(t)}$$
$$u_2'(t) = \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}$$

$\longrightarrow$

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}\,dt + c_1$$
$$u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}\,dt + c_2$$

Specific Solution

$$y = u_1(t)y_1(t) + u_2(t)y_2(t)$$

Variation of Parameters

Theorem Consider the **non-homogeneous** differential equation

$$y'' + p(t)y' + q(t)y = g(t),$$

where p, q, g are continuous on an open interval $I = (a, b)$. There is always a particular solution $S(t)$ which can be expressed as

$$S(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1, y_2](s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1, y_2](s)}\,ds,$$

where $\{y_1, y_2\}$ is a fundamental solution set for the homogeneous problem and $t_0 \in I$. The general solution is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + S(t).$$