Differential Equations

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Higher Order Equations

System of Equations

Systems of ODEs & Higher Order ODEs

Higher-Order ODEs

$$y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)}), \quad a \leq t \leq b$$

$$y(a) = \alpha_1, y'(a) = \alpha_2, \quad \ldots \quad y^{(m-1)}(a) = \alpha_m$$

Example: Newton's Second Law (F=ma)

$$v = \frac{dx}{dt}, \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$ma = m\frac{d^2x}{dt^2}, \quad F = -\gamma\frac{dx}{dt} - kx$$

$$F = ma, \quad m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - kx$$

$$z(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}, \quad z_0 = \begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

$$\frac{dx}{dt} = v, \quad m\frac{dv}{dt} = -\gamma v - kx \quad \rightarrow \quad \frac{dx}{dt} = v, \quad \frac{dv}{dt} = -\frac{\gamma}{m}v - \frac{k}{m}x$$

$$\rightarrow \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} \rightarrow \frac{dz}{dt} = \underline{\underline{A}}\,z, \quad z(0) = z_0$$

Systems of ODEs & Higher Order ODEs

Higher-Order ODEs

$$y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)}), \quad a \leq t \leq b$$
$$y(a) = \alpha_1, y'(a) = \alpha_2, \quad \ldots \quad y^{(m-1)}(a) = \alpha_m$$

System of ODEs

$$\frac{du_1}{dt} = f_1(t, u_1, u_2, \ldots, u_m),$$
$$\frac{du_2}{dt} = f_2(t, u_1, u_2, \ldots, u_m),$$
$$\vdots$$
$$\frac{du_m}{dt} = f_m(t, u_1, u_2, \ldots, u_m),$$

$$u_1(a) = \alpha_1, \ u_2(a) = \alpha_2, \ \ldots, \ u_m(a) = \alpha_m$$

System for Higher-Order ODEs

$$\frac{du_1}{dt} = \frac{dy}{dt} = u_2,$$
$$\frac{du_2}{dt} = \frac{dy'}{dt} = u_3,$$
$$\vdots$$
$$\frac{du_{m-1}}{dt} = \frac{dy^{(m-2)}}{dt} = u_m$$
$$\frac{du_m}{dt} = \frac{dy^{(m-1)}}{dt} = y^{(m)} = f(t, y, y', \ldots, y^{(m-1)})$$
$$= f(t, u_1, u_2, \ldots, u_m)$$

$$u_1(a) = y(a) = \alpha_1, \quad u_2(a) = y'(a) = \alpha_2,$$
$$\ldots \quad u_m(a) = y^{(m-1)}(a) = \alpha_m$$

Nonlinear Differential Equations

Behaviors in Higher Dimensions

(optional materials)

Lipschitz Condition
Non-Lipschitz Equations
Lorenz System
Chaotic Dynamics

Lipschitz Continuity for Functions of R^n

Definition: A function $f(t, y_1, \ldots, y_m)$ is called Lipschitz if for some constant L

$$|f(t, u_1, \ldots, u_m) - f(t, z_1, \ldots, z_m)| \leq L \sum_{j=1}^{m} |u_j - z_j|$$

for all $(t, u_1, \ldots, u_m)$ and $(t, z_1, \ldots, z_m)$ in D, where

$$D = \{(t, u_1, \ldots, u_m) \mid a \leq t \leq b \text{ and } -\infty < u_i < \infty, \text{ for each } i = 1, 2, \ldots, m\}$$

System of ODEs

$$\frac{du_1}{dt} = f_1(t, u_1, u_2, \ldots, u_m),$$

$$\frac{du_2}{dt} = f_2(t, u_1, u_2, \ldots, u_m),$$

$$\vdots$$

$$\frac{du_m}{dt} = f_m(t, u_1, u_2, \ldots, u_m),$$

$$u_1(a) = \alpha_1, \; u_2(a) = \alpha_2, \; \ldots, \; u_m(a) = \alpha_m$$

Example: $f(t,u_1,u_2) = u_1 u_2$ **not** Lipschitz!

Example: $f(t,u_1,u_2) = tu_1 + tu_2$ is Lipschitz.

Well-posedness of System of ODEs

System of ODEs

$$\frac{du_1}{dt} = f_1(t, u_1, u_2, \ldots, u_m),$$
$$\frac{du_2}{dt} = f_2(t, u_1, u_2, \ldots, u_m),$$
$$\vdots$$
$$\frac{du_m}{dt} = f_m(t, u_1, u_2, \ldots, u_m),$$

$$u_1(a) = \alpha_1,\ u_2(a) = \alpha_2,\ \ldots,\ u_m(a) = \alpha_m$$

Definition: A function $f(t, y_1, \ldots, y_m)$ is called Lipschitz if for some constant L

$$|f(t, u_1, \ldots, u_m) - f(t, z_1, \ldots, z_m)| \leq L \sum_{j=1}^{m} |u_j - z_j|$$

for all $(t, u_1, \ldots, u_m)$ and $(t, z_1, \ldots, z_m)$ in D, where

$$D = \{(t, u_1, \ldots, u_m) \mid a \leq t \leq b \text{ and } -\infty < u_i < \infty, \text{ for each } i = 1, 2, \ldots, m\}$$

Theorem: If the functions f_i in the system of ODEs each satisfy the Lipschitz condition on D then

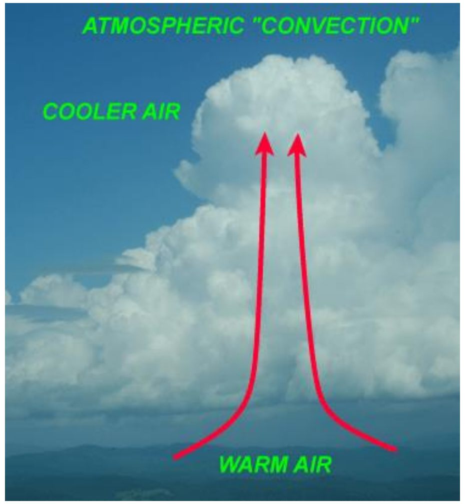
(1) there exists a solution
(2) the solution is unique
(3) robust to perturbations in initial conditions.

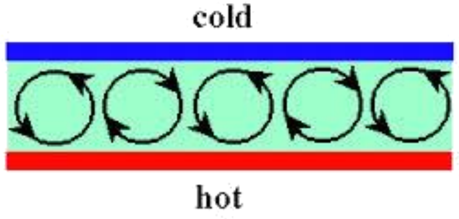
Key Points:

Establishes ODE is well-posed, if Lipschitz.

What behaviors can arise if NOT Lipschitz?

Lorenz System





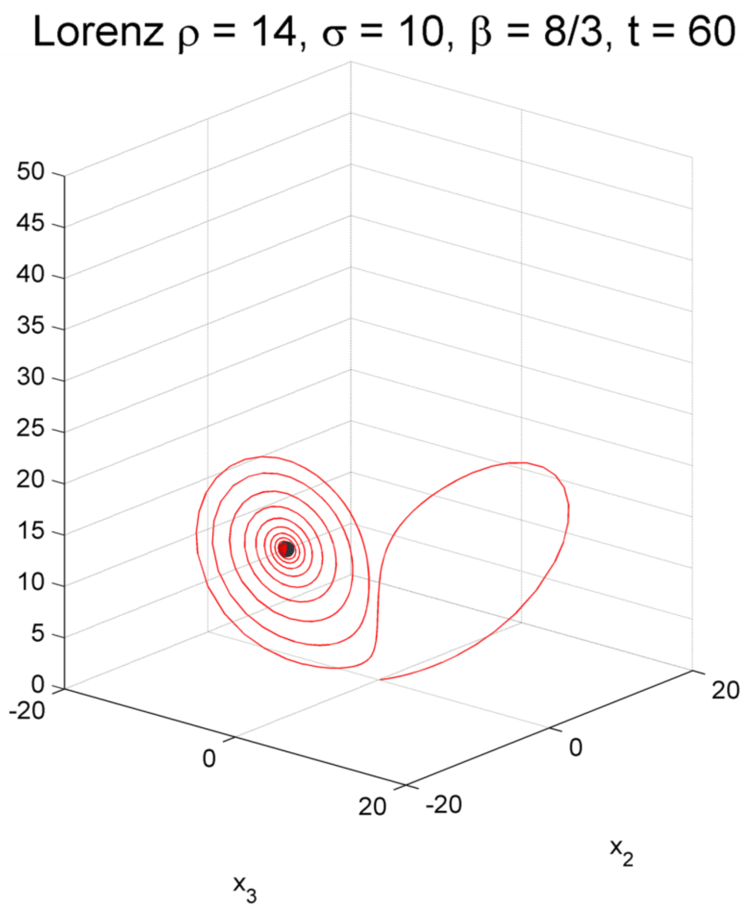
Simplified model of convection:

Lorenz Model (System of ODEs)

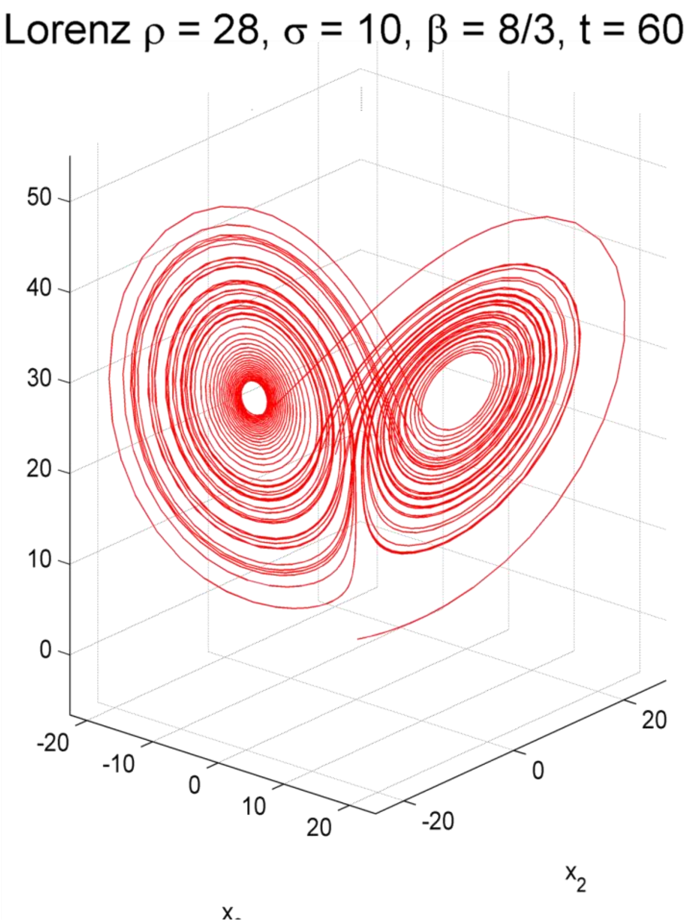
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y,$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z.$$

The $xy - \beta z$ term is non-linear and not Lipschitz.

Lorenz System

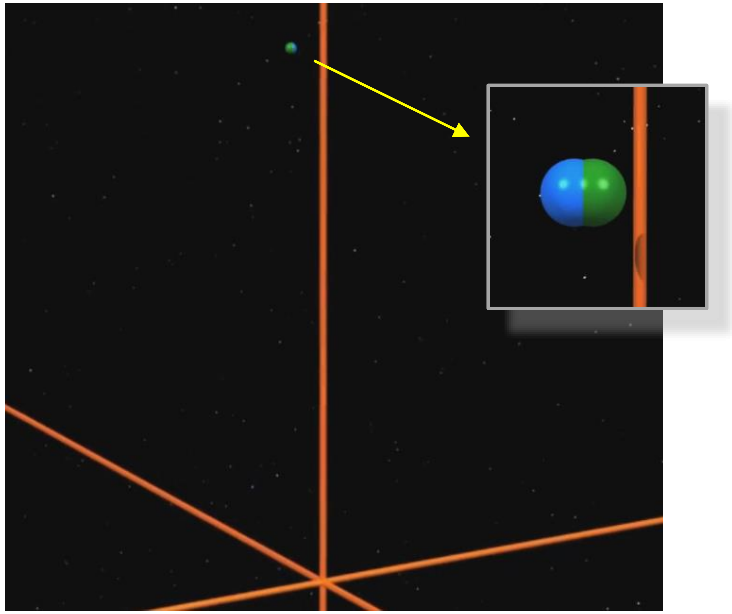


stable limit



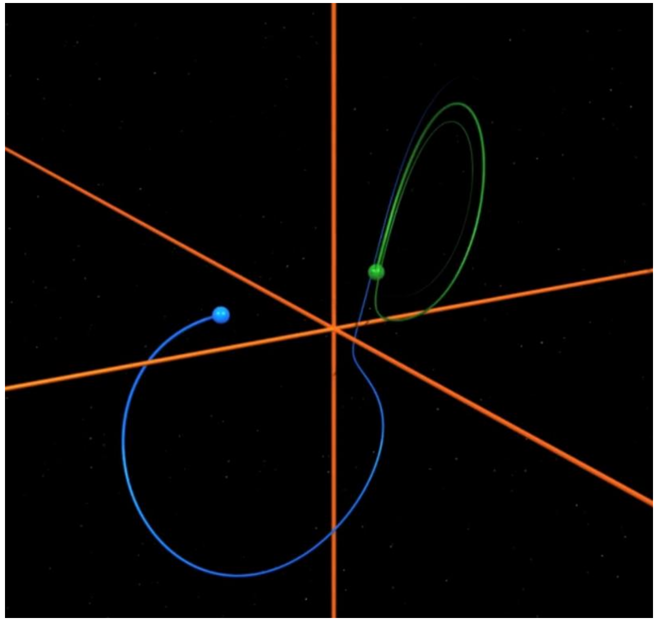
chaotic

Lorenz System



Leys 2013

initial conditions
(close together)



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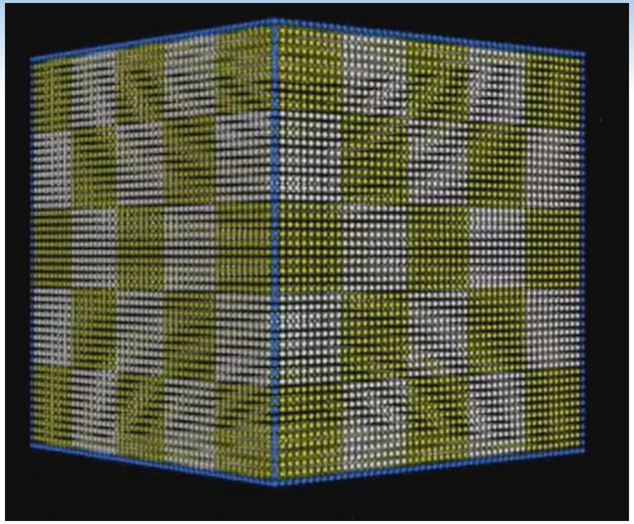
trajectories diverge

Initial conditions:

- extremely sensitive to location of initial conditions.
- given a start location can not predict the future precisely.

However, the future locations are restricted to be near subset of points!

Lorenz System



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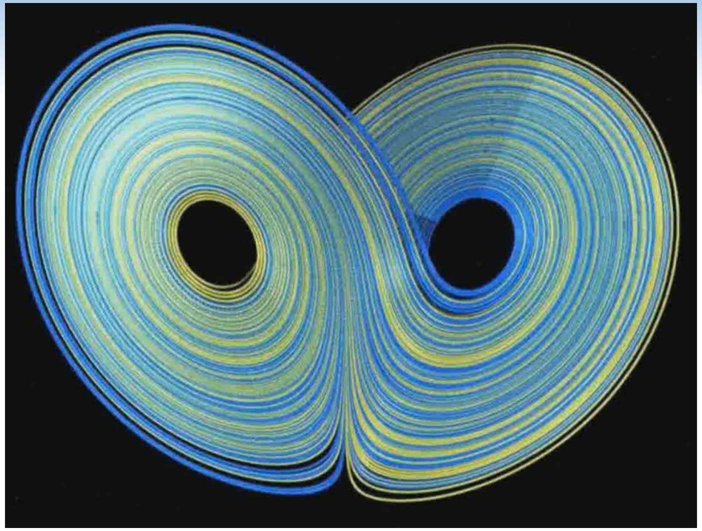
many initial conditions
(ensemble)



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attractor

Trajectories:
- almost all initial conditions move toward subset.
- called an “attractor” (stable to perturbations).
- dynamics after transient is only on the attractor.
- structure? dimensionality? topology?
- need to study dynamics at ensemble level.

Summary:
- Lipschitz continuity is very important to help ensure well-posed models!
- Ensemble-level studies helpful in gaining further insights.

Linear Differential Equations

Higher Order Systems

Solution Techniques

Higher Order Equations

Consider the n^{th}-order differential equation

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_n(t)\, y = g(t) \qquad (*)$$

$$L[y] = \left(\frac{d^n}{dt^n} + p_1(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + p_n(t)\right) y, \quad L[y] = g(t)$$

initial value problem

$$y(0) = y_0^{(0)}, \ \frac{dy}{dt}(0) = y_0^{(1)}, \ \ldots, \ \frac{d^{n-1}y}{dt^{n-1}}(0) = y_0^{(n-1)}$$

Theorem: If $p_1(t), p_2(t), \ldots, p_n(t)$ are continuous on $I = (a,b)$ then there exists exactly one solution $y(t) = \phi(t)$ of $(*)$. For the initial value problem $y(t)$ exists throughout the interval I.

A possible strategy to obtain solutions is to use superposition principle.

Candidate solutions based of $y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$, where $\{y_1, y_2, \ldots, y_n\}$ is a fundamental solution set.

Higher Order Equations

Ex: $\frac{dy_1}{dt} = -2y_1,\ y_1(0) = 3 \quad \rightarrow \quad y_1(t) = c_1 e^{-2t},\ y_1(0) = 3 \Rightarrow c_1 = 3 \Rightarrow y_1(t) = 3e^{-2t}$

$\frac{dy_2}{dt} = -3y_2,\ y_2(0) = 4 \quad \rightarrow \quad y_2(t) = c_2 e^{-3t},\ y_2(0) = 4 \Rightarrow c_2 = 4 \Rightarrow y_2(t) = 4e^{-3t}.$

$$\frac{d\underline{y}}{dt} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\underline{y},\quad \underline{y}_0 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Ex: $\frac{d\underline{y}}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\underline{y},\ \underline{y}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$ $\quad \frac{dy_1}{dt} = y_1,\ y_1(0) = 2,\quad \underline{z} = P\underline{y},\ \underline{y} = P^{-1}\underline{z}$

$\frac{dy_2}{dt} = y_1 - y_2,\ y_2(0) = 1,\quad \frac{d\underline{z}}{dt} = P\frac{d\underline{y}}{dt},\ \frac{d\underline{y}}{dt} = P^{-1}\frac{d\underline{z}}{dt}$

$P^{-1}\frac{d\underline{z}}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}P^{-1}\underline{z},\ \rightarrow \frac{d\underline{z}}{dt} = P\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}P^{-1}\underline{z}$, try to choose P so that $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ is diagonalized.

$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\underline{v} = \lambda\underline{v},\quad p(\lambda) = \det\left(\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} - \lambda I\right) = \det\begin{bmatrix} 1-\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) - 0 = -(\lambda-1)(\lambda+1)$

$p(\lambda) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = -1$

$\begin{bmatrix} 1-\lambda_1 & 0 \\ 1 & -1-\lambda_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \Rightarrow \underline{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \checkmark,\quad \begin{bmatrix} 1-\lambda_2 & 0 \\ 1 & -1-\lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \checkmark$

Higher Order Equations

Ex: $\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$

$\frac{dy_1}{dt} = y_1, \quad y_1(0) = 2,$

$\frac{dy_2}{dt} = y_1 - y_2, \quad y_2(0) = 1,$

$\mathbf{z} = P\mathbf{y}, \quad \boxed{\mathbf{y} = P^{-1}\mathbf{z}}$

$\frac{d\mathbf{z}}{dt} = P\frac{d\mathbf{y}}{dt}, \quad \frac{d\mathbf{y}}{dt} = P^{-1}\frac{d\mathbf{z}}{dt}$

$P^{-1}\frac{d\mathbf{z}}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}P^{-1}\mathbf{z}, \quad \rightarrow \quad \frac{d\mathbf{z}}{dt} = P\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}P^{-1}\mathbf{z},$ try to choose P so that $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ is diagonalized.

$\boxed{\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\mathbf{v} = \lambda\mathbf{v},}$

$p(\lambda) = \det\left(\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} - \lambda I\right) = \det\begin{bmatrix} 1-\lambda & 0 \\ 1 & -1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) - 0 = -(\lambda-1)(\lambda+1)$

$p(\lambda) = 0 \Rightarrow \lambda_1 = 1, \ \lambda_2 = -1$

$\begin{bmatrix} 1-\lambda_1 & 0 \\ 1 & -1-\lambda_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \Rightarrow \boxed{\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}} \checkmark, \quad \begin{bmatrix} 1-\lambda_2 & 0 \\ 1 & -1-\lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \boxed{\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \checkmark$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\mathbf{y}(t) = z_1(t)\mathbf{v}_1 + z_2(t)\mathbf{v}_2 = P^{-1}\mathbf{z}, \quad P^{-1} = [\mathbf{v}_1 | \mathbf{v}_2] = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad P = \frac{1}{\det(P^{-1})}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$

$\frac{d\mathbf{y}}{dt} = \frac{dz_1}{dt}\mathbf{v}_1 + \frac{dz_2}{dt}\mathbf{v}_2 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}(z_1(t)\mathbf{v}_1 + z_2(t)\mathbf{v}_2)$

$= z_1(t)\mathbf{v}_1 - z_2(t)\mathbf{v}_2 \Rightarrow \left(\frac{dz_1}{dt} - z_1\right)\mathbf{v}_1 + \left(\frac{dz_2}{dt} + z_2\right)\mathbf{v}_2 = 0$

by linear independence

$\Rightarrow \frac{dz_1}{dt} - z_1 = 0, \quad \frac{dz_2}{dt} + z_2 = 0 \Rightarrow z_1(t) = c_1 e^{t}, \quad z_2(t) = c_2 e^{-t}$

$\mathbf{y}(t) = c_1 e^{t}\mathbf{v}_1 + c_2 e^{-t}\mathbf{v}_2,$

Higher Order Equations

Consider nth-order equation:

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Def: For a collection of solutions {y1,y2,...yn} the **Wronskian** is

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Def: A collection of functions {f1,f2,...fn} are called **Linearly Independent** if

$$k_1 f_1(t) + k_2 f_2(t) + \cdots + k_n f_n(t) = 0 \longrightarrow k_1 = k_2 = \cdots = k_n = 0$$

Homogeneous equation:

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0.$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

General solution:

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

Determining coefficients:

$$c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = y_0$$

$$c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) = y_0'$$

$$\vdots$$

$$c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = y_0^{(n-1)}$$

Theorem:
Consider the homogeneous equation with $p_1, p_2, \ldots, p_n$ where each is continuous on the interval $I = (a, b)$. Let $y_1, y_2, \ldots, y_n$ each be solutions of the differential equation. If the Wronskian $W = W[y_1, y_2, \ldots, y_n](t_0) \neq 0$ for some $t_0 \in I$ then $W \neq 0$ for all $t \in I$ and any solution of the differential equation can be expressed as $y(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$.

Higher Order Equations

Ex:

$y''' + y' = 0, \quad y_1(t) = 1, \quad y_2(t) = \cos(t), \quad y_3(t) = \sin(t), \quad y(0) = 2, \quad y'(0) = 3, \quad y''(0) = -1$

Wronskian

$$W[y_1, y_2, y_3](t) = \det\begin{bmatrix} 1 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{bmatrix} = \sin^2(t) - (-\cos^2(t)) = \sin^2(t) + \cos^2(t) = 1.$$

$t_0 = 0$:

$$W(t_0) = \det\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = (1)\det\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = (1)(0 - (-1)) = 1 \checkmark$$

Thm $\Rightarrow$ $y(t) = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t)$

$= c_1 + c_2\cos(t) + c_3\sin(t).$

Initial Conditions

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \Rightarrow c_1 = 1,\ c_2 = 1,\ c_3 = 3 \Rightarrow$$

Solution

$y(t) = 1 + \cos(t) + 3\sin(t).$

Higher Order Equations

Ex: $y''' + 2y'' - y' - 2y = 0$, $y_1(t) = e^{t}$, $y_2(t) = e^{-t}$, $y_3(t)\ e^{-3t}$, $y(0) = 4$, $y'(0) = -2$, $y''(0) = 12$

Wronskian

$$t_0 = 0,\quad W[y_1, y_2, y_3](t_0) = \det\begin{bmatrix} y_1(t_0) & y_2(t_0) & y_3(t_0) \\ y_1'(t_0) & y_2'(t_0) & y_3'(t_0) \\ y_1''(t_0) & y_2''(t_0) & y_3''(t_0) \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 9 \end{bmatrix}$$

$$= (1)\det\begin{bmatrix} -1 & -3 \\ 1 & 9 \end{bmatrix} + (-1)\det\begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix} + (1)\det\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$$

$$= -9 + 3 - 9 + 1 + (-3) + 1 = -18 + 2 = -16 \neq 0$$

thm $\Rightarrow$ $y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{-3t}$

initial conditions

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 9 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 12 \end{bmatrix} \Rightarrow c_1 = 2,\ c_2 = 1,\ c_3 = 1 \Rightarrow y(t) = 2e^{t} + e^{-t} + e^{-3t}.$$

Ex: Are the following functions linearly independent

$f_1(t)=1,\ f_2(t)=t,\ f_3(t)=\frac{1}{2}t^2,\ -\infty<t<\infty$

$k_1 f_1(t)+k_2 f_2(t)+k_3 f_3(t)=0,\quad t_0=0,\ t_1=1,\ t_2=-1$

$k_1 f_1(0)+k_2 f_2(0)+k_3 f_3(0)=0$

$k_1 f_1(1)+k_2 f_2(1)+k_3 f_3(1)=0$

$k_1 f_1(-1)+k_2 f_2(-1)+k_3 f_3(-1)=0$

$$\underbrace{\begin{bmatrix} f_1(t_0) & f_2(t_0) & f_3(t_0) \\ f_1(t_1) & f_2(t_1) & f_3(t_1) \\ f_1(t_2) & f_2(t_2) & f_3(t_2) \end{bmatrix}}_{A}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}=0$$

if $\det(A)\neq 0 \Rightarrow k_1=k_2=k_3=0.$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & \frac{1}{2} \\ 1 & -1 & \frac{1}{2} \end{bmatrix}}_{A}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\det(A)=(1)\det\begin{bmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}=\frac{1}{2}-\left(-\frac{1}{2}\right)=1\neq 0$$

$\Rightarrow k_1=k_2=k_3=0.$

$$\begin{cases} k_1=0 \\ k_1+k_2+\frac{1}{2}k_3=0 \\ k_1-k_2+\frac{1}{2}k_3=0 \end{cases} \Rightarrow$$

$$=\begin{cases} k_1=0 \\ k_2+\frac{1}{2}k_3=0 \\ k_3=0 \end{cases} = \begin{cases} k_1=0 \\ k_2=0 \\ k_3=0 \end{cases}$$