Differential Equations

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Higher Order Linear Equations

Consider nth-order equation:

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Def: For a collection of solutions {y1,y2,...yn} the **Wronskian** is

$$W(y_1,\ldots,y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Def: A collection of functions {f1,f2,...fn} are called **Linearly Independent** if

$$k_1 f_1(t) + k_2 f_2(t) + \cdots + k_n f_n(t) = 0 \longrightarrow k_1 = k_2 = \cdots = k_n = 0$$

Homogeneous equation:

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0.$$

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

General solution:

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

Determining coefficients:

$$c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = y_0$$
$$c_1 y'_1(t_0) + \cdots + c_n y'_n(t_0) = y'_0$$
$$\vdots$$
$$c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = y_0^{(n-1)}$$

Theorem:
Consider the homogeneous equation with $p_1, p_2, \ldots, p_n$ where each is continuous on the interval $I = (a, b)$. Let $y_1, y_2, \ldots, y_n$ each be solutions of the differential equation. If the Wronskian $W = W[y_1, y_2, \ldots, y_n](t_0) \neq 0$ for some $t_0 \in I$ then $W \neq 0$ for all $t \in I$ and any solution of the differential equation can be expressed as $y(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$.

Higher Order Equations

Ex:

$y''' + y' = 0, \quad y_1(t) = 1, \quad y_2(t) = \cos(t), \quad y_3(t) = \sin(t), \quad y(0) = 2, \quad y'(0) = 3, \quad y''(0) = -1$

Wronskian

$$W[y_1, y_2, y_3](t) = \det \begin{bmatrix} 1 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{bmatrix} = \sin^2(t) - (-\cos^2(t)) = \sin^2(t) + \cos^2(t) = 1.$$

$t_0 = 0$:

$$W(t_0) = \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = (1)\det \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = (1)(0-(-1)) = 1 \checkmark$$

Thm $\Rightarrow$ $y(t) = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t)$

$= c_1 + c_2 \cos(t) + c_3 \sin(t).$

Initial Conditions

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \Rightarrow c_1 = 1, \ c_2 = 1, \ c_3 = 3 \Rightarrow$$

Solution

$y(t) = 1 + \cos(t) + 3\sin(t).$

Higher Order Equations

Ex! $y''' + 2y'' - y' - 2y = 0$, $y_1(t) = e^{t}$, $y_2(t) = e^{-t}$, $y_3(t)\, e^{-3t}$, $y(0) = 4$, $y'(0) = -2$, $y''(0) = 12$

Wronskian

$t_0 = 0$, $W[y_1, y_2, y_3](t_0) = \det \begin{bmatrix} y_1(t_0) & y_2(t_0) & y_3(t_0) \\ y_1'(t_0) & y_2'(t_0) & y_3'(t_0) \\ y_1''(t_0) & y_2''(t_0) & y_3''(t_0) \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 9 \end{bmatrix}$

$= (1)\det\begin{bmatrix} -1 & -3 \\ 1 & 9 \end{bmatrix} + (-1)\det\begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix} + (1)\det\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$

$= -9 + 3 - 9 + 1 + (-3) + 1 = -18 + 2 = -16 \neq 0$

thm $\Rightarrow$ $y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{-3t}$

initial conditions

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 12 \end{bmatrix} \Rightarrow$ $c_1 = 2$, $c_2 = 1$, $c_3 = 1$ $\Rightarrow$ $y(t) = 2e^{t} + e^{-t} + e^{-3t}$.

Higher Order Equations

Ex: Are the following functions linearly independent

$f_1(t) = 1,\ f_2(t) = t,\ f_3(t) = \frac{1}{2}t^2,\ -\infty < t < \infty$

$k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) = 0, \quad t_0 = 0,\ t_1 = 1,\ t_2 = -1$

$$k_1 f_1(0) + k_2 f_2(0) + k_3 f_3(0) = 0$$
$$k_1 f_1(1) + k_2 f_2(1) + k_3 f_3(1) = 0$$
$$k_1 f_1(-1) + k_2 f_2(-1) + k_3 f_3(-1) = 0$$

$$\underbrace{\begin{bmatrix} f_1(t_0) & f_2(t_0) & f_3(t_0) \\ f_1(t_1) & f_2(t_1) & f_3(t_1) \\ f_1(t_2) & f_2(t_2) & f_3(t_2) \end{bmatrix}}_{A} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0$$

if $\det(A) \neq 0 \Rightarrow k_1 = k_2 = k_3 = 0.$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & \frac{1}{2} \\ 1 & -1 & \frac{1}{2} \end{bmatrix}}_{A} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\det(A) = (1)\det\begin{bmatrix} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1 \neq 0$$

$\Rightarrow k_1 = k_2 = k_3 = 0.$

$$\left\{\begin{array}{l} k_1 = 0 \\ k_1 + k_2 + \frac{1}{2}k_3 = 0 \\ k_1 - k_2 + \frac{1}{2}k_3 = 0 \end{array}\right\} \Rightarrow$$

$$= \left\{\begin{array}{l} k_1 = 0 \\ k_2 + \frac{1}{2}k_3 = 0 \\ k_3 = 0 \end{array}\right\} = \left\{\begin{array}{l} k_1 = 0 \\ k_2 = 0 \\ k_3 = 0 \end{array}\right\}$$

Higher Order Equations & Systems of Equations

Existence, Uniqueness, Robustness

Higher Order ODEs & Systems of ODEs

Summary: Higher-Order ODEs and Systems of ODEs

Higher-Order ODEs

$$y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)}), \quad a \le t \le b$$
$$y(a) = \alpha_1, y'(a) = \alpha_2, \ \ldots \ y^{(m-1)}(a) = \alpha_m$$

System of ODEs

$$\frac{du_1}{dt} = f_1(t, u_1, u_2, \ldots, u_m),$$
$$\frac{du_2}{dt} = f_2(t, u_1, u_2, \ldots, u_m),$$
$$\vdots$$
$$\frac{du_m}{dt} = f_m(t, u_1, u_2, \ldots, u_m),$$

$$u_1(a) = \alpha_1, \ u_2(a) = \alpha_2, \ \ldots, \ u_m(a) = \alpha_m$$

System for Higher-Order ODEs

$$\frac{du_1}{dt} = \frac{dy}{dt} = u_2,$$
$$\frac{du_2}{dt} = \frac{dy'}{dt} = u_3,$$
$$\vdots$$
$$\frac{du_{m-1}}{dt} = \frac{dy^{(m-2)}}{dt} = u_m$$
$$\frac{du_m}{dt} = \frac{dy^{(m-1)}}{dt} = y^{(m)} = f(t, y, y', \ldots, y^{(m-1)})$$
$$= f(t, u_1, u_2, \ldots, u_m)$$
$$u_1(a) = y(a) = \alpha_1, \quad u_2(a) = y'(a) = \alpha_2, \quad \cdots \ u_m(a) = y^{(m-1)}(a) = \alpha_m$$

Well-posedness

Lipschitz condition: A function $f(t, y_1, \ldots, y_m)$ is called Lipschitz if for some constant L

$$|f(t, u_1, \ldots, u_m) - f(t, z_1, \ldots, z_m)| \le L \sum_{j=1}^{m} |u_j - z_j|$$

for all $(t, u_1, \ldots, u_m)$ and $(t, z_1, \ldots, z_m)$ in D, where

$$D = \{(t, u_1, \ldots, u_m) \mid a \le t \le b \text{ and } -\infty < u_i < \infty, \text{ for each } i = 1, 2, \ldots, m\}$$

Robustness to perturbations

For any $\epsilon > 0$, consider the perturbation to the ODE

$$\tilde{\alpha} = \alpha + \delta_0, \quad \tilde{f}_i = f_i + \delta_i(t),$$

where $|\delta_0| < \delta, |\delta_i(t)| < \delta$. Let $\tilde{y}$ be the solution to the ODE with $\tilde{\alpha}$ and $\tilde{f}_i$.

We say the solution is **robust to perturbations** if there is a $\delta > 0$ so that $|\tilde{y}(t) - y(t)| < \epsilon$.

Theorem (systems of ODEs): If the functions $\mathbf{f}_i$ in the system of ODEs each satisfy the **Lipschitz condition** on D then
(i) there exists a solution
(ii) the solution is unique
(iii) robust to perturbations (initial conditions, rhs).

Corollary (higher-order ODEs): If the functions **f** of the ODE satisfies the **Lipschitz condition** on D then
(i) there exists a solution
(ii) the solution is unique
(iii) robust to perturbations (initial conditions, rhs).

Lipschitz Continuity for Functions of R^n

Definition: A function $f(t, y_1, \ldots, y_m)$ is called Lipschitz if for some constant L

$$|f(t,u_1,\ldots,u_m) - f(t,z_1,\ldots,z_m)| \le L\sum_{j=1}^{m} |u_j - z_j|$$

for all $(t,u_1,\ldots,u_m)$ and $(t,z_1,\ldots,z_m)$ in D, where

$$D = \{(t,u_1,\ldots,u_m) \mid a \le t \le b \text{ and } -\infty < u_i < \infty, \text{ for each } i = 1,2,\ldots,m\}$$

System of ODEs

$$\frac{du_1}{dt} = f_1(t,u_1,u_2,\ldots,u_m),$$

$$\frac{du_2}{dt} = f_2(t,u_1,u_2,\ldots,u_m),$$

$$\vdots$$

$$\frac{du_m}{dt} = f_m(t,u_1,u_2,\ldots,u_m),$$

$$u_1(a) = \alpha_1,\ u_2(a) = \alpha_2,\ \ldots,\ u_m(a) = \alpha_m$$

Example: $f(t,u_1,u_2) = u_1u_2$ **not** Lipschitz!

$|u_1u_2 - z_1z_2| \overset{?}{\le} L(|u_1-z_1| + |u_2-z_2|)$

let $z_1 = u_1$, then for any L, we have

$|u_1u_2 - u_1z_2| = |u_1||u_2-z_2| > L|u_2-z_2|$

for $|u_1| > L$. $\Rightarrow$ not Lipschitz.

Example: $f(t,u_1,u_2) = tu_1 + tu_2$ is Lipschitz.

$a \le t \le b$, let $L = \max\{a,b\}$, then

$|tu_1 + tu_2 - (tz_1 + tz_2)| = |t||(u_1-z_1) + (u_2-z_2)|$

$\le |t|(|u_1-z_1| + |u_2-z_2|)$

$\le L(|u_1-z_1| + |u_2-z_2|)$ ✓

$\Rightarrow$ f is Lipschitz.

Higher Order Linear Equations

Undetermined Coefficients

Variation of Parameters

Higher Order Linear Equations

Consider nth-order equation:

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Homogeneous equation:

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0.$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

$\longrightarrow$ **General solution:**

$$y = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)$$

Inhomogeneous equation:

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

$\longrightarrow$ **General solution:**

$$y = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) + S(t) \qquad L[S] = g(t).$$

Theorem: Consider the linear equation with $p_1(t), p_2(t), \ldots, p_n(t)$ where each is continuous on the interval $I = (a, b)$. The solution $y(t)$ exists on the interval I and is unique.

Theorem: Consider the homogeneous equation with $p_1, p_2, \ldots, p_n$ where each is continuous on the interval $I = (a, b)$. Let $y_1, y_2, \ldots, y_n$ each be solutions of the differential equation. If the Wronskian $W = W[y_1, y_2, \ldots, y_n](t_0) \neq 0$ for some $t_0 \in I$ then $W \neq 0$ for all $t \in I$ and any solution of the differential equation can be expressed as $y(t) = c_1y_1(t) + \cdots + c_ny_n(t)$.

Higher Order Linear Equations

Method of Undetermined Coefficients

Higher Order Linear Equations: Constant Coefficients

Homogeneous equation with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0,$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Candidate solution

$\tilde{y}(t) = e^{rt}$

$L[y] = a_0 r^n e^{rt} + a_1 r^{n-1} e^{rt} + \cdots + a_{n-1} r e^{rt} + a_n e^{rt} = 0$

$\Rightarrow \underbrace{p(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n}_{\text{characteristic polynomial}} = 0$

If the roots of $p(r) = 0$ are all real and they are are distinct then

$y_1(t) = e^{r_1 t}$, $y_2(t) = e^{r_2 t}$, ..., $y_n(t) e^{r_n t}$.

The general solution

$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$.

General solution:

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

Determining coefficients:

$$\begin{aligned} c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\ c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\ &\vdots \\ c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}$$

Higher Order Linear Equations: Constant Coefficients

Ex:
$$\left\{\begin{array}{l} y^{(4)} + y^{(3)} - 7y^{(2)} - y^{(1)} + 6y = 0 \\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -2,\ y'''(0) = -1 \end{array}\right\}$$

$$p(r) = r^4 + r^3 - 7r^2 - r + 6 = 0$$

$$r_1 = 1,\ r_2 = -1,\ r_3 = -2,\ r_4 = -3$$

$$y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{-2t} + c_4 e^{-3t}$$ ← general solution

Now we need to find c_1, c_2, c_3, c_4 so $y(t)$ matches the initial conditions.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & -8 & -27 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} \Rightarrow \underline{c} = \begin{bmatrix} \frac{11}{8} \\ \frac{5}{12} \\ -\frac{2}{3} \\ -\frac{1}{8} \end{bmatrix} \Rightarrow c_1 = \frac{11}{8},\ c_2 = \frac{5}{12},\ c_3 = \frac{-2}{3},\ c_4 = \frac{-1}{8}$$

$$\Rightarrow y(t) = \frac{11}{8} e^{t} + \frac{5}{12} e^{-t} + \frac{-2}{3} e^{-2t} + \frac{-1}{8} e^{-3t}.$$

Higher Order Linear Equations

Method of Undetermined Coefficients

Real and Distinct Roots

Higher Order Linear Equations: Constant Coefficients (Real and Distinct Roots)

Homogeneous equation with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0.$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Summary of Solution Technique:

Find roots of the **characteristic polynomial**:

$$\rho(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

Are all the roots real and distinct?

Yes, then use **general solution** form

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$$

Find coefficients using initial conditions by solving

$$\begin{aligned} c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\ c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\ &\vdots \\ c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}$$

No, then need to use another method.

Higher Order Linear Equations: Constant Coefficients

Inhomogeneous equation with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t)$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

$\longrightarrow$

General solution:

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + S(t)$$

$$L[S] = g(t).$$

Determining coefficients:

$$c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = y_0 - S(t_0)$$

$$c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) = y_0' - S'(t_0)$$

$$\vdots$$

$$c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = y_0^{(n-1)} - S^{(n-1)}(t_0)$$

Higher Order Linear Equations: Constant Coefficients (Real and Distinct Roots)

Ex: $y^{(3)} - 2y^{(2)} - y' + 2y = 3e^t$

$y(0) = 5,\ y'(0) = 3,\ y''(0) = 9$

$p(r) = r^3 - 2r^2 - r + 2 = 0$

$p(r) = (r^2-1)(r-2) = (r-1)(r+1)(r-2)$

$r_1 = 1,\ r_2 = -1,\ r_3 = 2$

$y_1(t) = e^t,\ y_2(t) = e^{-t},\ y_3(t) = e^{2t}$

$\Rightarrow \boxed{y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + s(t)}$

$L[s] = 3e^t,\quad s(t) = (A_1 + A_2 t)e^t$

$s'(t) = A_2 e^t + (A_1 + A_2 t)e^t = (A_1 + A_2)e^t + A_2 t e^t$

$s''(t) = A_2 e^t + A_2 e^t + (A_1 + A_2 t)e^t = (A_1 + 2A_2)e^t + A_2 t e^t = -3e^t + -\frac{3}{2} t e^t \Big|_{t=0} = -3$

$s^{(3)}(t) = A_2 e^t + A_2 e^t + A_2 e^t + (A_1 + A_2 t)e^t = (A_1 + 3A_2)e^t + A_2 t e^t$

$e^t(A_1 + 3A_2 - 2A_1 - 4A_2 - A_1 - A_2 + 2A_1)$

$+ te^t(A_2 - 2A_2 - A_2 + 2A_2) = 3e^t$

$(0\cdot A_1 - 2A_2)e^t + (0\cdot A_1 + 0\cdot A_2)te^t = 3e^t + 0\cdot te^t$

$\Rightarrow -2A_2 = 3,\quad 0 = 0 \Rightarrow A_2 = \frac{-3}{2}.$

$\Rightarrow s(t) = \frac{-3}{2} t e^t.\quad s'(0) = \frac{-3}{2},$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 - 0 \\ 3 - (-\frac{3}{2}) \\ 9 - (-3) \end{bmatrix}.$$

Higher Order Linear Equations: Constant Coefficients (Real and Distinct Roots)

Inhomogeneous equation with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t)$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Summary of Solution Technique:

Find roots of the **characteristic polynomial**:

$$\rho(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

Are all the roots real and distinct?

Yes, then use **general solution** form

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t} + S(t)$$

Find special solution solving $L[S] = g(t)$.

Find coefficients using initial conditions by solving

$$c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = y_0 - S(t_0)$$

$$c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) = y_0' - S'(t_0)$$

$$\vdots$$

$$c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = y_0^{(n-1)} - S^{(n-1)}(t_0)$$

No, then need to use another method.

Higher Order Linear Equations

Method of Undetermined Coefficients

Distinct Real and Complex Roots (possibly mixed)

Higher Order Linear Equations: Constant Coefficients (Distinct Roots, Real and Complex-Valued)

Homogeneous equation with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0.$$

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Summary of Solution Technique:

Find roots of the **characteristic polynomial**:

$$\rho(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

Are the roots distinct real and complex-valued?

Yes, then use **general solution** form

$$\begin{aligned} y(t) &= c_1 \exp(t\lambda_1)\cos(\mu_1 t) + c_2 \exp(t\lambda_1)\sin(\mu_1 t) + \cdots + c_{2k-1}\exp(t\lambda_k)\cos(\mu_k t) + c_{2k}\exp(t\lambda_k)\sin(\mu_k t) \\ &+ c_{2k+1}\exp(t r_{2k+1}) + \cdots c_n \exp(t r_n). \end{aligned}$$

Find coefficients using initial conditions by solving

$$\begin{aligned} c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\ c_1 y'_1(t_0) + \cdots + c_n y'_n(t_0) &= y'_0 \\ &\vdots \\ c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}$$

No, then need to use another method.

Lemma: Roots of a polynomial p(r) with real coefficients have any complex-valued roots occurring in conjugate pairs $r_+ = \lambda + i\mu, \quad r_- = \lambda - i\mu.$

Exponentials:

$$\exp(tr_-) = \exp((\lambda - i\mu)t) = e^{\lambda t}\cos(\mu t) - i e^{\lambda t}\sin(\mu t)$$

$$\exp(tr_+) = \exp((\lambda + i\mu)t) = e^{\lambda t}\cos(\mu t) + i e^{\lambda t}\sin(\mu t)$$

Lemma: For the two complex-valued roots r-, r+, we have any function which is a linear combination of the complex exponentials can be expressed in terms of real-valued functions as

$$\begin{aligned} \tilde{y}(t) &= \tilde{c}_+ \exp(tr_+) + \tilde{c}_- \exp(tr_-) \\ &= c_+ \exp(t\lambda)\cos(\mu t) + c_- \exp(t\lambda)\sin(\mu t) \end{aligned}$$

Higher Order Linear Equations: Constant Coefficients (Complex Roots)

Ex: $y^{(3)} + y = 0$

$y(0) = 2,\ y'(0) = -1,\ y''(0) = 1 - \frac{\pi^2}{9}.$

$p(r) = r^3 + 1 = 0, \quad e^{i\phi} = \cos(\phi) + i\sin(\phi)$

$r^3 = -1, \quad -1 = e^{i\pi}, \quad r = e^{i\theta}, \quad 1 = e^{i2\pi \cdot n}$

$e^{i3\theta} = e^{i\pi} \Rightarrow 3\theta \equiv \pi \mod 2\pi$

$3\theta_1 = \pi, \quad 3\theta_2 = \pi - 2\pi, \quad 3\theta_3 = \pi + 2\pi$

$\theta_1 = \frac{\pi}{3}, \quad \theta_2 = -\frac{\pi}{3}, \quad \theta_3 = \pi$

$r_1 = e^{i\frac{\pi}{3}}, \quad r_2 = e^{-i\frac{\pi}{3}}, \quad r_3 = e^{i\pi} = -1$

$r_1 = \cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3}), \quad r_2 = \cos(\frac{\pi}{3}) - i\sin(\frac{\pi}{3}), \quad r_3 = -1$

General Solution

$$\boxed{y(t) = c_1 e^{\lambda_1 t}\cos(\mu_1 t) + c_2 e^{\lambda_1 t}\sin(\mu_1 t) + c_3 e^{-t}}.$$

$\lambda_1 = \cos(\frac{\pi}{3})$

$\mu_1 = \sin(\frac{\pi}{3})$

$$\begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & \mu_1 & -1 \\ \lambda_1^2 - \mu_1^2 & 2\lambda_1\mu_1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 - \frac{\pi^2}{9} \end{bmatrix}$$

$$y'(t) = c_1\lambda_1 e^{\lambda_1 t}\cos(\mu_1 t) - c_1 e^{\lambda_1 t}\mu_1 \sin(\mu_1 t) + c_2\lambda_1 e^{\lambda_1 t}\sin(\mu_1 t) + c_2 e^{\lambda_1 t}\mu_1\cos(\mu_1 t) - c_3 e^{-t}$$

$$y''(t) = c_1\lambda_1^2 e^{\lambda_1 t}\cos(\mu_1 t) - c_1\lambda_1\mu_1 e^{\lambda_1 t}\sin(\mu_1 t) - c_1\lambda_1\mu_1 e^{\lambda_1 t}\sin(\mu_1 t) - c_1 e^{\lambda_1 t}\mu_1^2\cos(\mu_1 t) + c_2\lambda_1^2 e^{\lambda_1 t}\sin(\mu_1 t) + c_2\lambda_1\mu_1 e^{\lambda_1 t}\cos(\mu_1 t) + c_2\lambda_1\mu_1 e^{\lambda_1 t}\cos(\mu_1 t) - c_2 e^{\lambda_1 t}\mu_1^2\sin(\mu_1 t) + c_3 e^{-t}.$$

Inhomogeneous equation with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = g(t)$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Lemma: Roots of a polynomial p(r) with real coefficients have any complex-valued roots occurring in conjugate pairs $r_+ = \lambda + i\mu, \quad r_- = \lambda - i\mu.$

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$$\begin{aligned} \tilde{y}(t) &= \tilde{c}_+ \exp(tr_+) + \tilde{c}_- \exp(tr_-) \\ &= c_+ \exp(t\lambda)\cos(\mu t) + c_- \exp(t\lambda)\sin(\mu t) \end{aligned}$$

Summary of Solution Technique:

Find roots of the **characteristic polynomial**:

$$\rho(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

Are the roots distinct real and complex-valued?

Yes, then use **general solution** form

$$\begin{aligned} y(t) &= c_1 \exp(t\lambda_1)\cos(\mu_1 t) + c_2 \exp(t\lambda_1)\sin(\mu_1 t) + \cdots + c_{2k-1}\exp(t\lambda_k)\cos(\mu_k t) + c_{2k}\exp(t\lambda_k)\sin(\mu_k t) \\ &+ c_{2k+1}\exp(tr_{2k+1}) + \cdots c_n \exp(tr_n) + S(t) \end{aligned}$$

Find coefficients using initial conditions by solving

$$c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = y_0 - S(t_0)$$
$$c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) = y_0' - S'(t_0)$$
$$\vdots$$
$$c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = y_0^{(n-1)} - S^{(n-1)}(t_0)$$

No, then need to use another method.

Higher Order Linear Equations

Method of Undetermined Coefficients

Repeated Roots

Higher Order Linear Equations: Constant Coefficients (Real, Complex, and Repeated Roots)

Homogeneous equation with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0.$$

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Summary of Solution Technique:

Are coefficients of the ODE constant?

Yes, then **find roots** of the **characteristic polynomial**:

$$p(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

Use **general solution** form

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

where yj determined from the cases discussed.

Find coefficients using initial conditions by solving

$$\begin{aligned} c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\ c_1 y'_1(t_0) + \cdots + c_n y'_n(t_0) &= y'_0 \\ &\vdots \\ c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}$$

No, must use another method.

Lemma: Roots r* of a polynomial p(r) with multiplicity s correspond to solutions

$$e^{r_1 t}, \quad t e^{r_1 t}, \quad t^2 e^{r_1 t}, \quad \ldots, \quad t^{s-1} e^{r_1 t}$$

Real-valued Case: roots with multiplicity s contribute functions yj

$$e^{r_1 t}, \quad t e^{r_1 t}, \quad t^2 e^{r_1 t}, \quad \ldots, \quad t^{s-1} e^{r_1 t}$$

Complex-valued Case: roots with multiplicity s contribute functions yj

$$e^{\lambda t}\cos\mu t, \quad e^{\lambda t}\sin\mu t, \quad t e^{\lambda t}\cos\mu t, \quad t e^{\lambda t}\sin\mu t,$$
$$\ldots, \quad t^{s-1} e^{\lambda t}\cos\mu t, \quad t^{s-1} e^{\lambda t}\sin\mu t.$$

Higher Order Linear Equations: Constant Coefficients

Ex: $y^{(3)} + 3y^{(2)} + 3y^{(1)} + y = 0$

$p(r) = r^3 + 3r^2 + 3r + 1 = 0$

$r = -1$ is root multiplicity 3

$p(r) = (r+1)^3$

$y(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t}$

Higher Order Linear Equations

Method of Undetermined Coefficients

Summary

Higher Order Linear Equations: Constant Coefficients (Summary)

Homogeneous equation with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0.$$
$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Summary of Solution Technique:

Are coefficients of the ODE constant?

Yes, then **find roots** of the **characteristic polynomial**:

$$\rho(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

Use **general solution** form

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Real-valued Case: roots with multiplicity s contribute functions yj

$$e^{r_1 t}, \quad te^{r_1 t}, \quad t^2 e^{r_1 t}, \quad \ldots, \quad t^{s-1} e^{r_1 t}$$

Complex-valued Case: roots with multiplicity s contribute functions yj

$$e^{\lambda t}\cos\mu t, \quad e^{\lambda t}\sin\mu t, \quad te^{\lambda t}\cos\mu t, \quad te^{\lambda t}\sin\mu t,$$
$$\ldots, \quad t^{s-1}e^{\lambda t}\cos\mu t, \quad t^{s-1}e^{\lambda t}\sin\mu t.$$

Higher Order Linear Equations: Constant Coefficients (Summary)

Inhomogeneous equation with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = g(t)$$
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Summary of Solution Technique:

Are coefficients of the ODE constant?

Yes, then **find roots** of the **characteristic polynomial**:

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$$e^{r_1 t}, \quad te^{r_1 t}, \quad t^2 e^{r_1 t}, \quad \ldots, \quad t^{s-1} e^{r_1 t}$$

Complex-valued Case: roots with multiplicity s contribute functions yj

$$e^{\lambda t}\cos \mu t, \quad e^{\lambda t}\sin \mu t, \quad te^{\lambda t}\cos \mu t, \quad te^{\lambda t}\sin \mu t,$$
$$\ldots, \quad t^{s-1}e^{\lambda t}\cos \mu t, \quad t^{s-1}e^{\lambda t}\sin \mu t.$$