Differential Equations

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Higher Order Linear Equations

Method of Undetermined Coefficients

Summary

Higher Order Linear Equations: Constant Coefficients (Summary)

Homogeneous equation with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0,$$
$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Summary of Solution Technique:

Are coefficients of the ODE constant?

Yes, then **find roots** of the **characteristic polynomial**:

$$p(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

Use **general solution** form

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

where yj determined from the cases discussed.

Find coefficients using initial conditions by solving

$$\begin{aligned} c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\ c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\ &\vdots \\ c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}$$

No, must use another method.

Lemma: Roots of a polynomial p(r) with real coefficients have any complex-valued roots occurring in conjugate pairs $r_+ = \lambda + i\mu, \quad r_- = \lambda - i\mu.$

Exponentials:

$$\exp(tr_-) = \exp((\lambda - i\mu)t)$$
$$\exp(tr_+) = \exp((\lambda + i\mu)t)$$

Lemma: For the two complex-valued roots r-, r+, we have any function which is a linear combination of the complex exponentials can be expressed in terms of real-valued functions as

$$\begin{aligned} \tilde{y}(t) &= \tilde{c}_+ \exp(tr_+) + \tilde{c}_- \exp(tr_-) \\ &= c_+ \exp(t\lambda)\cos(\mu t) + c_- \exp(t\lambda)\sin(\mu t) \end{aligned}$$

Lemma: Roots r* of a polynomial p(r) with multiplicity s correspond to solutions

$$e^{r_1 t}, \quad te^{r_1 t}, \quad t^2 e^{r_1 t}, \quad \ldots, \quad t^{s-1} e^{r_1 t}$$

Real-valued Case: roots with multiplicity s contribute functions yj

$$e^{r_1 t}, \quad te^{r_1 t}, \quad t^2 e^{r_1 t}, \quad \ldots, \quad t^{s-1} e^{r_1 t}$$

Complex-valued Case: roots with multiplicity s contribute functions yj

$$e^{\lambda t}\cos\mu t, \quad e^{\lambda t}\sin\mu t, \quad te^{\lambda t}\cos\mu t, \quad te^{\lambda t}\sin\mu t,$$
$$\ldots, \quad t^{s-1}e^{\lambda t}\cos\mu t, \quad t^{s-1}e^{\lambda t}\sin\mu t.$$

Higher Order Linear Equations: Constant Coefficients (Summary)

Inhomogeneous equation with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t)$$
$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Summary of Solution Technique:

Are coefficients of the ODE constant?

Yes, then **find roots** of the **characteristic polynomial**:

$$\rho(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

Use **general solution** form

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + S(t)$$

where yj determined from the cases discussed.

Find coefficients using initial conditions by solving

$$c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = y_0 - S(t_0)$$
$$c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) = y_0' - S'(t_0)$$
$$\vdots$$
$$c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = y_0^{(n-1)} - S^{(n-1)}(t_0)$$

No, must use another method.

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Lemma: Roots r* of a polynomial p(r) with multiplicity s correspond to solutions

$$e^{r_1 t}, \quad te^{r_1 t}, \quad t^2 e^{r_1 t}, \quad \ldots, \quad t^{s-1} e^{r_1 t}$$

Real-valued Case: roots with multiplicity s contribute functions yj

$$e^{r_1 t}, \quad te^{r_1 t}, \quad t^2 e^{r_1 t}, \quad \ldots, \quad t^{s-1} e^{r_1 t}$$

Complex-valued Case: roots with multiplicity s contribute functions yj

$$e^{\lambda t}\cos\mu t, \quad e^{\lambda t}\sin\mu t, \quad te^{\lambda t}\cos\mu t, \quad te^{\lambda t}\sin\mu t, \quad \ldots, \quad t^{s-1}e^{\lambda t}\cos\mu t, \quad t^{s-1}e^{\lambda t}\sin\mu t.$$

Higher Order Linear Equations

Method of Variation of Parameters

Variation of Parameters for Higher Order Linear Equations

Inhomogeneous Differential Equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

Solutions to Homogeneous Equation

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

Candidate Form of Solution

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t)\ , \quad L[Y] = g(t)$$

First Derivative

$$Y' = (u_1 y_1' + u_2 y_2' + \cdots + u_n y_n') + (u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n) \rightarrow 0$$

Imposed condition

$$u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0$$

Higher Order Derivatives

$$Y^{(m)} = u_1 y_1^{(m)} + u_2 y_2^{(m)} + \cdots + u_n y_n^{(m)}$$

Imposed conditions

$$u_1' y_1^{(m-1)} + u_2' y_2^{(m-1)} + \cdots + u_n' y_n^{(m-1)} = 0\ , \quad m = 1, 2, \ldots, n-1$$

nth-Order Derivative

$$Y^{(n)} = (u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)}) + (u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)})$$

$L[y_i] = 0$

Required condition

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g$$

Conditions for Coefficient Functions u

$$y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' = 0$$
$$y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' = 0$$
$$y_1'' u_1' + y_2'' u_2' + \cdots + y_n'' u_n' = 0$$
$$\vdots$$
$$y_1^{(n-1)} u_1' + \cdots + y_n^{(n-1)} u_n' = g$$

Solving linear system by Cramer's rule

$$u_m'(t) = \frac{g(t)W_m(t)}{W(t)}$$
$$m = 1, 2, \ldots, n$$

Solution

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)}\, ds$$

Definition: The $W_m(t)$ is obtained by replacing the mth-column of **W** by $[0,0,\ldots 1]^T$

Wronskian (Abel's Theorem)

$$W(t) = W(y_1, \ldots, y_n)(t) = c \exp\left[-\int p_1(t)\, dt\right]$$

Variation of Parameters

Ex: $y''' - y'' - y' + y = g(t)$

Find a special solution $S(t)$.

$y_1(t) = e^t, \quad y_2(t) = te^t, \quad y_3(t) = e^{-t}.$

$$S(t) = \sum_{m=1}^{3} y_m(t) \int_{t_0}^{t} \frac{g(s)\,W_m(s)}{W(s)}\,ds,$$

$W_m(s) = \det(Q_m(s))$, $Q_m(s) = Q(s)$ with the m^{th} column replaced by $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

$$W(s) = \underbrace{\begin{vmatrix} y_1(s) & y_2(s) & y_3(s) \\ y_1'(s) & y_2'(s) & y_3'(s) \\ y_1''(s) & y_2''(s) & y_3''(s) \end{vmatrix}}_{Q(s)} = \det(Q(s)),$$

Wronskian

$$W(t) = W[e^t, te^t, e^{-t}] = \begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & e^{-t} \end{vmatrix} = e^t \begin{vmatrix} 1 & t & 1 \\ 1 & (t+1) & -1 \\ 1 & (t+2) & 1 \end{vmatrix} = e^t \begin{vmatrix} 1 & t & 1 \\ 0 & 1 & -2 \\ 0 & 2 & 0 \end{vmatrix} = 4e^t.$$

$$W_1(t) = \begin{vmatrix} 0 & te^t & e^{-t} \\ 0 & (t+1)e^t & -e^{-t} \\ 1 & (t+2)e^t & e^{-t} \end{vmatrix} = -2t-1, \quad W_2(t) = \begin{vmatrix} e^t & 0 & e^{-t} \\ e^t & 0 & -e^{-t} \\ e^t & 1 & e^{-t} \end{vmatrix} = 2, \quad W_3(t) = \begin{vmatrix} e^t & te^t & 0 \\ e^t & (t+1)e^t & 0 \\ e^t & (t+2)e^t & 1 \end{vmatrix} = e^{2t}$$

$$S(t) = e^t \int_{t_0}^{t} \frac{(-2s-1)\,g(s)}{4e^s}\,ds + te^t \int_{t_0}^{t} \frac{2g(s)}{4e^s}\,ds + e^{-t} \int_{t_0}^{t} \frac{e^{2s} g(s)}{4e^s}\,ds = \frac{1}{4}\int_{t_0}^{t} \left(e^{t-s}\left[-1+2(t-s)\right] + e^{-(t-s)}\right) g(s)\,ds.$$

Systems of First Order Equations

Solution Techniques

Higher Order ODEs & Systems of ODEs: Summary

Higher-Order ODEs

$$y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)}), \quad a \le t \le b$$
$$y(a) = \alpha_1, y'(a) = \alpha_2, \quad \ldots \quad y^{(m-1)}(a) = \alpha_m$$

System of ODEs

$$\frac{du_1}{dt} = f_1(t, u_1, u_2, \ldots, u_m),$$
$$\frac{du_2}{dt} = f_2(t, u_1, u_2, \ldots, u_m),$$
$$\vdots$$
$$\frac{du_m}{dt} = f_m(t, u_1, u_2, \ldots, u_m),$$

$$u_1(a) = \alpha_1, \; u_2(a) = \alpha_2, \; \ldots, \; u_m(a) = \alpha_m$$

System for Higher-Order ODEs

$$\frac{du_1}{dt} = \frac{dy}{dt} = u_2,$$
$$\frac{du_2}{dt} = \frac{dy'}{dt} = u_3,$$
$$\vdots$$
$$\frac{du_{m-1}}{dt} = \frac{dy^{(m-2)}}{dt} = u_m$$
$$\frac{du_m}{dt} = \frac{dy^{(m-1)}}{dt} = y^{(m)} = f(t, y, y', \ldots, y^{(m-1)})$$
$$= f(t, u_1, u_2, \ldots, u_m)$$
$$u_1(a) = y(a) = \alpha_1, \quad u_2(a) = y'(a) = \alpha_2,$$
$$\ldots \; u_m(a) = y^{(m-1)}(a) = \alpha_m$$

Systems of First Order Equations

Linear Algebra Review

Linear Systems

Linear System of Equations

System with **n equations** and **n unknowns**

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Solve for x in Ax = b.

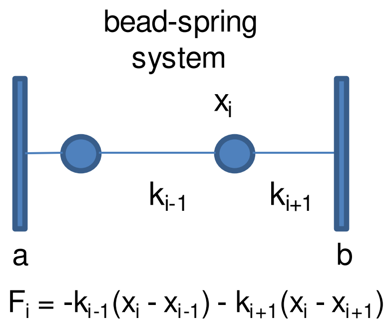
Need theory to determine when this is possible **(existence / uniqueness).**

Methods to solve

- algebriac approaches (gaussian elimination, factorizations)
- computational methods (direct methods, iterative methods)
- issues: tractability, robustness to small errors.

Example

For bead-spring system, find locations X_i of beads that balances the forces.



Linear Systems

Eigenvalues & Eigenvectors

Def: A vector $\underline{v}$ is called an eigenvector of a matrix A if there exists a scalar $\lambda \in \mathbb{R}$ so that

$$A\underline{v} = \lambda \underline{v}.$$

Def: The scalar $\lambda \in \mathbb{R}$ is called an eigenvalue of the matrix A.

Def: determinant 2x2 system

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Example

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\lambda_1 = -1, \lambda_2 = 3$ are eigenvalues, $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors.

Verify:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\underline{v}_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underbrace{(-1)}_{\lambda_1}\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\underline{v}_1}, \quad \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\underline{v}_2} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \underbrace{3}_{\lambda}\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\underline{v}_2}.$$

$$\underbrace{\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}}_{A - \lambda I,\ \lambda = 1}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

Characteristic Polynomial

$p(\lambda) = \det(A - \lambda I)$, each eigenvalue satisfies $p(\lambda) = 0$.

Remark: $p(\lambda) = \det(A - \lambda I) = 0 \iff \exists\, \underline{v}$ s.t. $(A - \lambda I)\underline{v} = 0 \iff A\underline{v} = \lambda\underline{v}$

Example $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $p(\lambda) = \det\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3,$

Linear Systems

Operations that preserve solution x:

(i) Row E_i can be multiplied by any non-zero constant γ, $E_i' \leftarrow \gamma E_i$.

(ii) Row E_j can be multiplied by any non-zero constant γ and added to row E_i

$E_i' \leftarrow E_i + \gamma E_j$

(iii) Rows E_i and E_j can always be transpositioned (exchanged) in order

$E_i' \leftarrow E_j$, $E_j' \leftarrow E_i$.

$A\underline{x} = b$

$a_{11}x_1 + a_{12}x_2 = b_1$
$a_{21}x_1 + a_{22}x_2 = b_2$

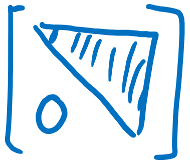
$A = \begin{bmatrix} E_1 \\ \hline E_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \hline b_2 \end{bmatrix}$

Def: The augmented matrix $[A|b] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$.

How can one use these operations to obtain a linear system that is easier to solve?

Linear Systems

We can try to put the matrix in an upper triangular form.
Gaussian Elimination Method.



Ex: $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -2 & 1 & 1 \\ 1 & -2 & -1 \end{array}\right] \xrightarrow{E_1 \leftrightarrow E_2} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 1 & 1 \end{array}\right] \xrightarrow{E_2' = E_2 + 2E_1} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & -3 & -1 \end{array}\right] \to \begin{Bmatrix} 1 \cdot x_1 - 2x_2 = -1 \\ 0 \cdot x_1 - 3x_2 = -1 \end{Bmatrix} \to \begin{array}{l} x_1 = -1 + \frac{2}{3} \\ x_2 = \frac{1}{3} \end{array}$$

Definition: A matrix is said to be in **row echelon form** if:
(i) The leading non-zero entry of each row is to the right of the leading non-zero entry of the rows above.
(ii) All non-zero rows are above the zero rows.

Definition: A matrix is in **reduced row echelon form** (canonical form) if the matrix is in row echelon form [(i) & (ii)] and
(iii) The leading non-zero entry is the only non-zero entry in its column (for non-zero rows).

Systems of First Order Equations

Theory

Systems of First Order Equations

First Order Linear Systems of Differential Equations

$$
\begin{aligned}
x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\
&\vdots \\
x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t)
\end{aligned}
$$

vector notation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

Homogeneous Case: Initial Value Problem

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \quad (*)$$
$$\mathbf{x}(t_0) = \boldsymbol{\xi}$$

Theorem

If the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system $(*)$, then the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution for any constants c_1 and c_2.

Theorem

If the vector functions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions of the system $(*)$ for each point in the interval $\alpha < t < \beta$, then each solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of the system $(*)$ can be expressed as a linear combination of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$

$$\boldsymbol{\phi}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$$

in exactly one way.

solutions

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \quad \ldots, \quad \mathbf{x}^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}$$

Systems of First Order Equations
Solution Techniques (Constant Coefficeints)

First Order Systems: Constant Coefficients

First Order Linear Systems with Constant Coefficients

$$x_1' = p_{11} \cdot x_1 + \cdots + p_{1n} \cdot x_n$$
$$\vdots$$
$$x_n' = p_{n1} \cdot x_1 + \cdots + p_{nn} \cdot x_n$$

→ vector notation

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

Solution Candidates

$$\mathbf{x} = \boldsymbol{\xi} e^{rt}, \longrightarrow r\boldsymbol{\xi} e^{rt} = \mathbf{A}\boldsymbol{\xi} e^{rt} \longrightarrow (\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$$

$$\det(\mathbf{A} - r\mathbf{I}) = 0.$$

Roots of the characteristic polynomial

$$\rho(r) = \det(\mathbf{A} - r\mathbf{I})$$

Cases

1. All eigenvalues are real and different from each other.
2. Some eigenvalues occur in complex conjugate pairs.
3. Some eigenvalues are repeated.