Differential Equations

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Systems of First Order Equations

Solution Techniques

Higher Order ODEs & Systems of ODEs: Summary

Higher-Order ODEs

$$y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)}), \quad a \leq t \leq b$$

$$y(a) = \alpha_1, y'(a) = \alpha_2, \quad \ldots \quad y^{(m-1)}(a) = \alpha_m$$

System of ODEs

$$\frac{du_1}{dt} = f_1(t, u_1, u_2, \ldots, u_m),$$

$$\frac{du_2}{dt} = f_2(t, u_1, u_2, \ldots, u_m),$$

$$\vdots$$

$$\frac{du_m}{dt} = f_m(t, u_1, u_2, \ldots, u_m),$$

$$u_1(a) = \alpha_1, \; u_2(a) = \alpha_2, \; \ldots, \; u_m(a) = \alpha_m$$

System for Higher-Order ODEs

$$\frac{du_1}{dt} = \frac{dy}{dt} = u_2,$$

$$\frac{du_2}{dt} = \frac{dy'}{dt} = u_3,$$

$$\vdots$$

$$\frac{du_{m-1}}{dt} = \frac{dy^{(m-2)}}{dt} = u_m$$

$$\frac{du_m}{dt} = \frac{dy^{(m-1)}}{dt} = y^{(m)} = f(t, y, y', \ldots, y^{(m-1)})$$

$$= f(t, u_1, u_2, \ldots, u_m)$$

$$u_1(a) = y(a) = \alpha_1, \quad u_2(a) = y'(a) = \alpha_2,$$

$$\cdots \; u_m(a) = y^{(m-1)}(a) = \alpha_m$$

Systems of First Order Equations

Linear Algebra Review

Linear Systems

Linear System of Equations

System with **n equations** and **n unknowns**

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

Solve for x in Ax = b.

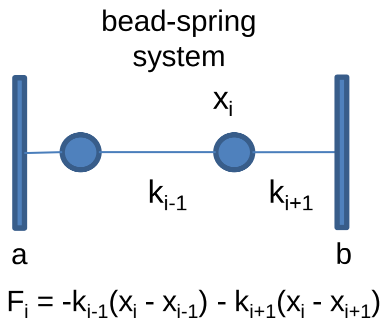
Need theory to determine when this is possible **(existence / uniqueness).**

Methods to solve

- algebriac approaches (gaussian elimination, factorizations)
- computational methods (direct methods, iterative methods)
- issues: tractability, robustness to small errors.

Example

For bead-spring system, find locations X_i of beads that balances the forces.



Eigenvalues & Eigenvectors

Def: A vector $\underline{v}$ is called an eigenvector of a matrix A if there exists a scalar $\lambda \in \mathbb{R}$ so that

$$A\underline{v} = \lambda \underline{v}.$$

Def: The scalar $\lambda \in \mathbb{R}$ is called an eigenvalue of the matrix A.

Def: determinant 2x2 system

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Example

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\lambda_1 = -1, \lambda_2 = 3$ are eigenvalues, $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors.

Verify:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\underline{v}_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underbrace{(-1)}_{\lambda_1}\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\underline{v}_1}, \quad \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\underline{v}_2} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \underbrace{3}_{\lambda}\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\underline{v}_2}.$$

$$\underbrace{\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}}_{A - \lambda I}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

$\lambda = 1$

Characteristic Polynomial

$p(\lambda) = \det(A - \lambda I)$, each eigenvalue satisfies $p(\lambda) = 0$.

Remark: $p(\lambda) = \det(A - \lambda I) = 0 \iff \exists\, \underline{v}$ s.t. $(A - \lambda I)\underline{v} = 0 \iff A\underline{v} = \lambda \underline{v}$

Example $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $p(\lambda) = \det\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3,$

Operations that preserve solution x:

(i) Row E_i can be multiplied by any non-zero constant γ, $E_i' \leftarrow \gamma E_i$.

(ii) Row E_j can be multiplied by any non-zero constant γ and added to row E_i

$E_i' \leftarrow E_i + \gamma E_j$

(iii) Rows E_i and E_j can always be transpositioned (exchanged) in order

$E_i' \leftarrow E_j, \quad E_j' \leftarrow E_i$.

$A\underline{x} = b$

$a_{11}x_1 + a_{12}x_2 = b_1$
$a_{21}x_1 + a_{22}x_2 = b_2$

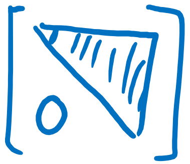
$A = \begin{bmatrix} E_1 \\ \hdashline E_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \hdashline b_2 \end{bmatrix}$

Def: The augmented matrix $[A|b] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$.

How can one use these operations to obtain a linear system that is easier to solve?

Linear Systems

We can try to put the matrix in an upper triangular form.
Gaussian Elimination Method.



Ex: $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -2 & 1 & 1 \\ 1 & -2 & -1 \end{array}\right] \xrightarrow{E_1 \leftrightarrow E_2} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 1 & 1 \end{array}\right] \xrightarrow{E_2' = E_2 + 2E_1} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & -3 & -1 \end{array}\right] \rightarrow \begin{Bmatrix} 1 \cdot x_1 - 2x_2 = -1 \\ 0 \cdot x_1 - 3x_2 = -1 \end{Bmatrix} \rightarrow \begin{array}{l} x_1 = -1 + \frac{2}{3} \\ x_2 = \frac{1}{3} \end{array}$$

Systems of First Order Equations

Theory

Systems of First Order Equations

First Order Linear Systems of Differential Equations

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned}$$

vector notation

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

Homogeneous Case: Initial Value Problem

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \quad (*)$$

$$\mathbf{x}(t_0) = \boldsymbol{\xi}$$

Theorem

If the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system $(*)$, then the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is also a solution for any constants c_1 and c_2.

Theorem

If the vector functions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions of the system $(*)$ for each point in the interval $\alpha < t < \beta$, then each solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of the system $(*)$ can be expressed as a linear combination of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$

$$\boldsymbol{\phi}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$$

in exactly one way.

solutions

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \quad \ldots, \quad \mathbf{x}^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}$$

Systems of First Order Equations

Solution Techniques (Constant Coefficients)

First Order Systems: Constant Coefficients

First Order Linear Systems with Constant Coefficients

$$x_1' = p_{11} \cdot x_1 + \cdots + p_{1n} \cdot x_n$$
$$\vdots$$
$$x_n' = p_{n1} \cdot x_1 + \cdots + p_{nn} \cdot x_n$$

→ vector notation

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

Solution Candidates

$$\mathbf{x} = \boldsymbol{\xi} e^{rt} \longrightarrow r\boldsymbol{\xi} e^{rt} = \mathbf{A}\boldsymbol{\xi} e^{rt} \longrightarrow (\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$$

$$\det(\mathbf{A} - r\mathbf{I}) = 0.$$

Roots of the characteristic polynomial

$$\rho(r) = \det(\mathbf{A} - r\mathbf{I})$$

Cases:

- **real and distinct** eigenvalues
- **complex-valued** eigenvalues
- **repeated** eigenvalues

- **mixture** of the cases above.

First Order Systems with Constant Coefficients (Real and Distinct Eigenvalues)

Homogeneous First Order System

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$
$$\mathbf{x}(t_0) = \boldsymbol{\xi}$$

Summary of Solution Technique:

Find roots of the **characteristic polynomial**:

$$\rho(r) = \det(\mathbf{A} - r\mathbf{I})$$

Are the roots distinct real?

Yes, then **fundamental solutions** are

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \quad \ldots, \quad \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)} e^{r_n t}$$

Use **solution of the form**

$$\mathbf{x} = c_1 \boldsymbol{\xi}^{(1)} e^{r_1 t} + \cdots + c_n \boldsymbol{\xi}^{(n)} e^{r_n t}$$

Find coefficients using initial conditions by solving

$$c_1 \mathbf{x}^{(1)}(t_0) + c_2 \mathbf{x}^{(2)}(t_0) + \cdots + c_n \mathbf{x}^{(n)}(t_0) = \mathbf{x}(t_0)$$

No, then need to use another method.

First Order Systems with Constant Coefficients (Real and Distinct Eigenvalues)

Ex: $\underline{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \underline{x}$

Find the general solution.

$p(r) = \det(A - rI), \quad A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$p(r) = \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3 = 0$

$r_1 = 3, \quad r_2 = -1, \quad \underline{x}^{(1)} = \underline{\xi}^{(1)} e^{3t}, \quad \underline{x}^{(2)} = \underline{\xi}^{(2)} e^{-t}.$

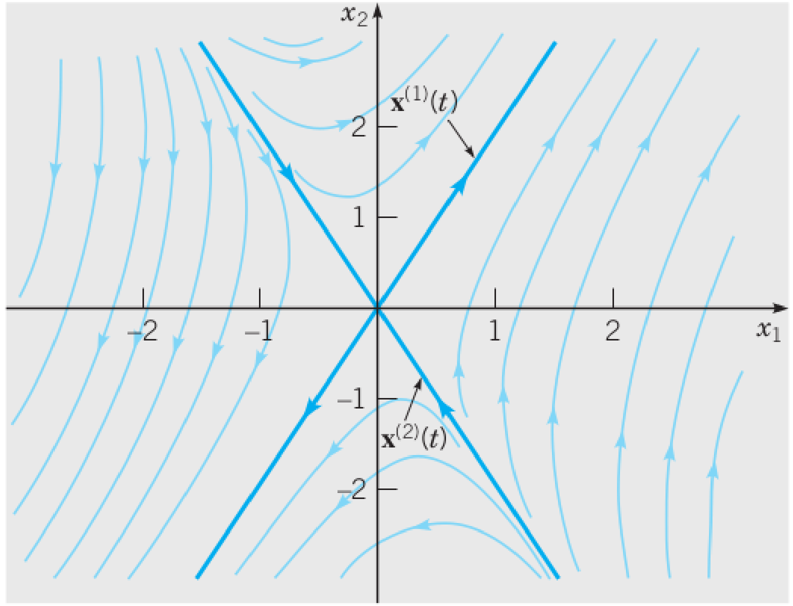
$(A - rI)\underline{\xi} = 0, \quad \begin{bmatrix} 1-r & 1 \\ 4 & 1-r \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$r_1 = 3: \quad \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftarrow \underline{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$r_2 = -1: \quad \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftarrow \underline{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$x^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}, \quad x^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$

$$x(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$



(a)

First Order Systems with Constant Coefficients (Complex-valued Eigenvalues)

Homogeneous First Order System

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$
$$\mathbf{x}(t_0) = \boldsymbol{\xi}$$

Summary of Solution Technique:

Find roots of the **characteristic polynomial**:

$$\rho(r) = \det(\mathbf{A} - r\mathbf{I})$$

Are the roots complex-valued?

Yes, then **fundamental solutions** are

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t)$$
$$\mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)$$

Use **solution of the form**

$$\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{v}(t)$$

Find coefficients using initial conditions by solving

$$c_1\mathbf{u}(t_0) + c_2\mathbf{v}(t_0) = \mathbf{x}(t_0)$$

No, then need to use another method.

Complex Eigenvalues $r = \lambda + i\mu$

$$(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{0} \qquad (\mathbf{A} - \overline{r}_1\mathbf{I})\overline{\boldsymbol{\xi}}^{(1)} = \mathbf{0}$$

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)}e^{r_1 t}, \qquad \mathbf{x}^{(2)}(t) = \overline{\boldsymbol{\xi}}^{(1)}e^{\overline{r}_1 t}$$

Euler Formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Formulation using Euler Formula $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$

$$\mathbf{x}^{(1)}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + ie^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)$$

$$\mathbf{x}^{(1)}(t) = \mathbf{u}(t) + i\mathbf{v}(t), \quad \mathbf{x}^{(2)}(t) = \mathbf{u}(t) - i\mathbf{v}(t)$$

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t)$$
$$\mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)$$

→ $\tilde{c}_1\mathbf{x}^{(1)} + \tilde{c}_2\mathbf{x}^{(2)} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t)$

First Order Systems: Constant Coefficients (Complex-Valued Eigenvalues)

Ex: $\underline{x}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \underline{x}$

Find the general solution.

$$p(r) = \begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = \left(-\tfrac{1}{2} - r\right)\left(-\tfrac{1}{2} - r\right) + 1$$

$$= \tfrac{1}{4} - r + r^2 + 1 = r^2 - r + \tfrac{5}{4} = 0$$

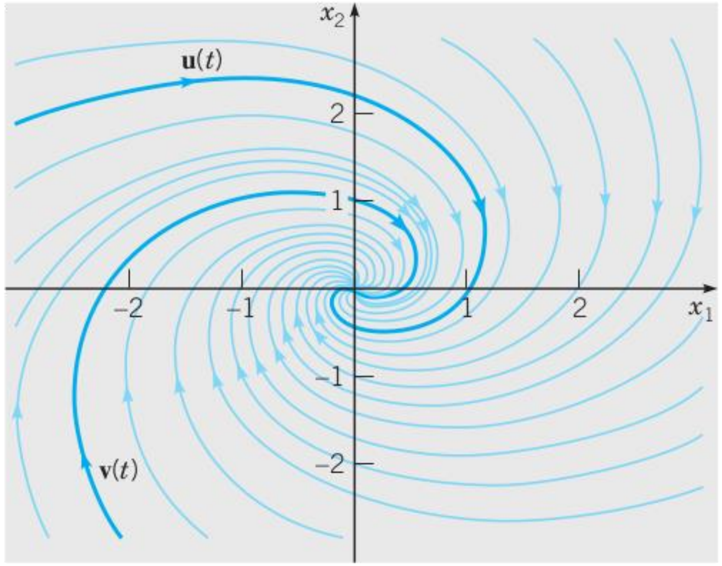
$r_1 = -\frac{1}{2} + i, \quad r_2 = -\frac{1}{2} - i$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \underline{0} \Leftarrow \underline{\xi}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \underline{a} + i\underline{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \underline{0} \Leftarrow \underline{\xi}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = \underline{a} - i\underline{b}$$

$$\underline{u} = e^{-\frac{1}{2}t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(t) - e^{-\frac{1}{2}t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(t) = e^{-\frac{1}{2}t}\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

$$\underline{v} = e^{-\frac{1}{2}t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(t) + e^{-\frac{1}{2}t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(t) = e^{-\frac{1}{2}t}\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$



$$\underline{x}(t) = c_1 e^{-\frac{1}{2}t}\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + c_2 e^{-\frac{1}{2}t}\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

First Order Systems with Constant Coefficients (Repeated Eigenvalues, s = 2, k=1)

Homogeneous First Order System

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$
$$\mathbf{x}(t_0) = \xi$$

Summary of Solution Technique:

Find roots of the **characteristic polynomial**:

$$\rho(r) = \det(\mathbf{A} - r\mathbf{I})$$

Are the roots repeated?

Yes, then **fundamental solutions** are

$$\mathbf{x}^{(1)} = \xi^{(1)} e^{rt},$$
$$\mathbf{x}^{(2)} = \eta^{(1)} e^{rt} + \xi^{(1)} t e^{rt}$$

Use **solution of the form**

$$\mathbf{x}(t) = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t)$$

Find coefficients using initial conditions by solving

$$c_1 \mathbf{x}^{(1)}(t_0) + c_2 \mathbf{x}^{(2)}(t_0) = \mathbf{x}(t_0).$$

No, then need to use another method.

Repeated Eigenvalues (s=2, k=1)

(i) find eigenvector for r, $\xi^{(1)}$ ← $(A - r\mathbf{I})\,\xi = 0$

(ii) construct the **generalized eigenvector** $\eta^{(1)}$ ← $(A - r\mathbf{I})^2\,\eta = 0$

can use

$$(A - r\mathbf{I})\,\eta = \xi$$
$$(A - r\mathbf{I})\,\xi = 0$$

(iii) this gives **fundamental solutions**

$$\mathbf{x}^{(1)} = \xi^{(1)} e^{rt},$$
$$\mathbf{x}^{(2)} = \eta^{(1)} e^{rt} + \xi^{(1)} t e^{rt}$$

s: algebraic multiplicity **k**: geometric multiplicity

First Order Systems: Constant Coefficients (Repeated Eigenvalues)

Ex: $\underline{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \underline{x}$

Find the general solution.

$p(r) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (1-r)(3-r) + 1 = r^2 - 4r + 4 = 0$

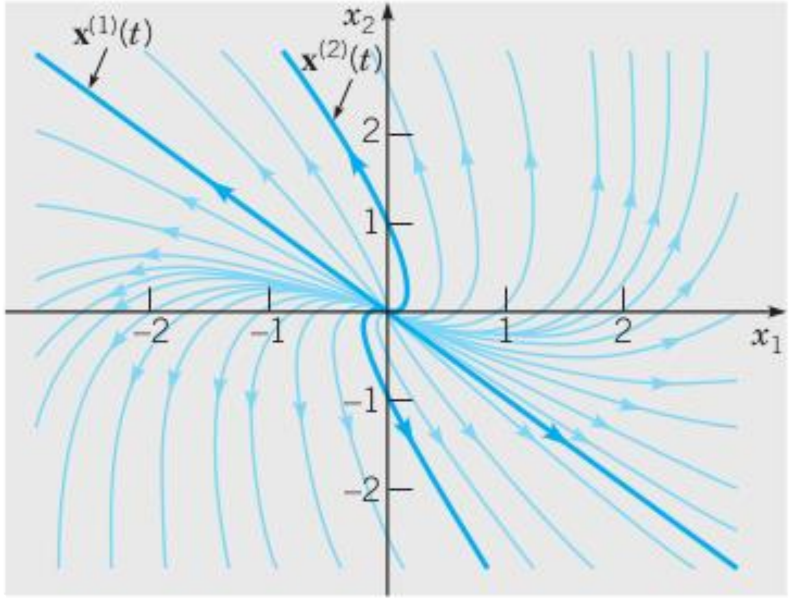
$r_1 = 2,\ r_2 = 2,$ algebraic multiplicity $s = 2$

$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \Leftarrow \underline{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ (A - rI)\underline{\xi} = 0$

$(A - rI)^2 \underline{\eta} = 0,\ \Leftarrow (A - rI)\underline{\eta} = \underline{\xi},$

$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Leftarrow \underline{\eta} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

$\underline{x}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t},\ \underline{x}^{(2)} = \underline{\eta} e^{2t} + \underline{\xi} t e^{2t} = \begin{bmatrix} -\frac{1}{2} + t \\ -\frac{1}{2} - t \end{bmatrix} e^{2t}$



$$\underline{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -\frac{1}{2} + t \\ -\frac{1}{2} - t \end{bmatrix} e^{2t}$$

First Order Systems with Constant Coefficients (Summary)

Homogeneous First Order System

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$
$$\mathbf{x}(t_0) = \xi$$

Summary of Solution Technique:

Are the coefficients constant?

Yes, then **find roots** of the **characteristic polynomial**:

$$\rho(r) = \det(\mathbf{A} - r\mathbf{I})$$

Construct the **fundamental solution set** using the cases

$$\{\mathbf{x}^{(1)}(t),\ \mathbf{x}^{(2)}(t),\ \ldots,\mathbf{x}^{(n)}(t)\}$$

Use **solution of the form**

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots c_n\mathbf{x}^{(n)}(t).$$

Find coefficients using initial conditions by solving

$$c_1\mathbf{x}^{(1)}(t_0) + c_2\mathbf{x}^{(2)}(t_0) + \cdots c_n\mathbf{x}^{(n)}(t_0) = \mathbf{x}(t_0)$$

No, then need to use another method.

Complex Eigenvalues $r = \lambda + i\mu$

$$(\mathbf{A} - r_1\mathbf{I})\xi^{(1)} = \mathbf{0} \qquad (\mathbf{A} - \overline{r}_1\mathbf{I})\overline{\xi}^{(1)} = \mathbf{0}$$

$$\mathbf{x}^{(1)}(t) = \xi^{(1)}e^{r_1 t}, \qquad \mathbf{x}^{(2)}(t) = \overline{\xi}^{(1)}e^{\overline{r}_1 t}$$

Formulation using Euler Formula $\xi^{(1)} = \mathbf{a} + i\mathbf{b}$

$$\mathbf{x}^{(1)}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + ie^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)$$

$$\mathbf{x}^{(1)}(t) = \mathbf{u}(t) + i\mathbf{v}(t), \quad \mathbf{x}^{(2)}(t) = \mathbf{u}(t) - i\mathbf{v}(t)$$

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t)$$
$$\mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)$$

→ $\tilde{c}_1\mathbf{x}^{(1)} + \tilde{c}_2\mathbf{x}^{(2)} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t)$

Repeated Eigenvalues (s=2,k=1)

(i) find eigenvector for r, $\xi^{(1)} \leftarrow (A - r\mathbf{I})\,\xi = 0$

(ii) construct the **generalized eigenvector** $\eta^{(1)} \leftarrow (A - r\mathbf{I})^2\,\eta = 0$

can use

$$(A - r\mathbf{I})\,\eta = \xi$$
$$(A - r\mathbf{I})\,\xi = 0$$

(iii) this gives **fundamental solutions**

$$\mathbf{x}^{(1)} = \xi^{(1)}e^{rt},$$
$$\mathbf{x}^{(2)} = \eta^{(1)}e^{rt} + \xi^{(1)}te^{rt}$$

s: algebraic multiplicity **k**: geometric multiplicity

Systems of First Order Equations

Phase Portraits

Dynamical Behaviors

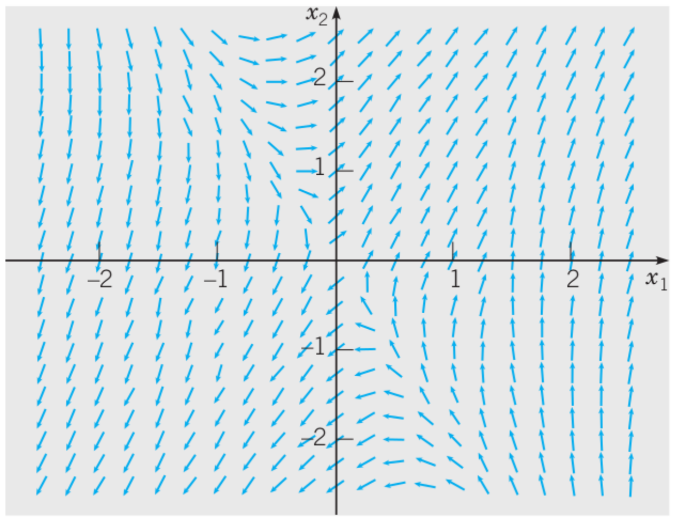
Phase Portraits

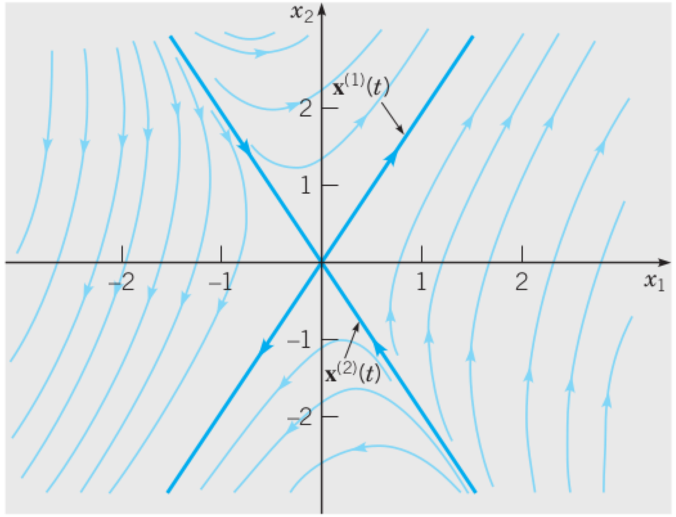
A **phase portrait** shows representative trajectories and vector field for the evolution of the state **x(t)** of the differential equation.

Ex:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

phase portrait





(a)

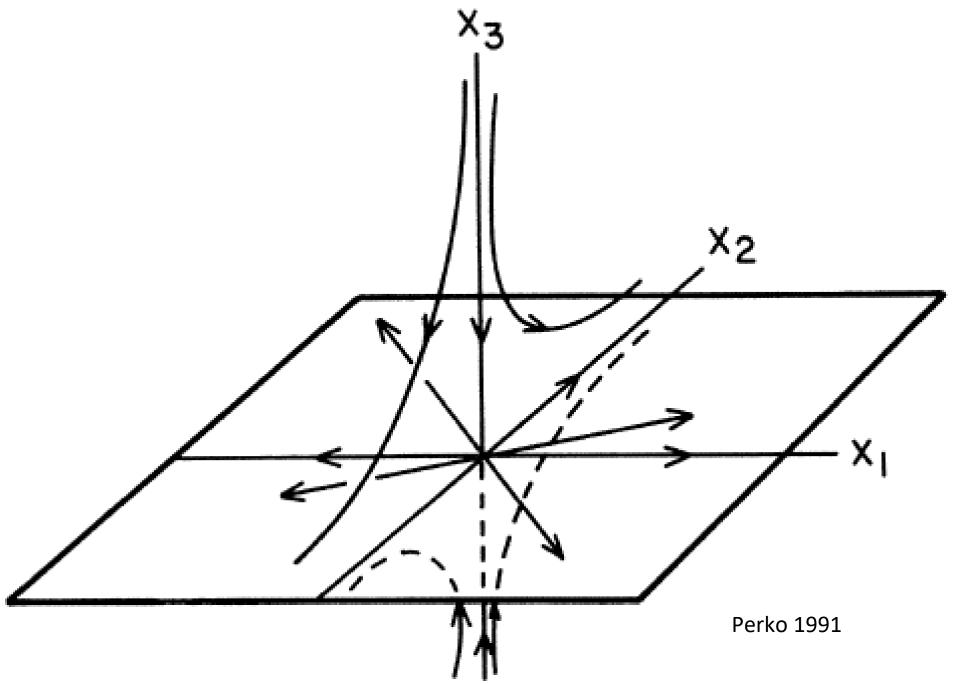
Ex:

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = x_2$$
$$\dot{x}_3 = -x_3$$

solution →

$$x_1(t) = c_1 e^{t}$$
$$x_2(t) = c_2 e^{t}$$
$$x_3(t) = c_3 e^{-t}$$

phase portrait



2D Linear Systems: Phase Portraits

Linear Dynamical System

$$\dot{\mathbf{x}} = A\mathbf{x}$$

Consider the following standard forms

$$\dot{\mathbf{x}} = B\mathbf{x}$$

with

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad \text{or} \quad B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

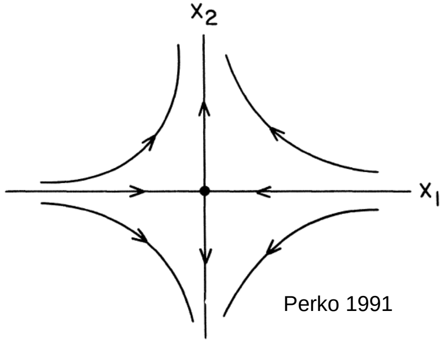
This has the solutions

$$\mathbf{x}(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \mathbf{x}_0 \,, \; \mathbf{x}(t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0 \,, \; \mathbf{x}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{x}_0$$

Case I:

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \text{ with } \lambda < 0 < \mu$$

phase portrait

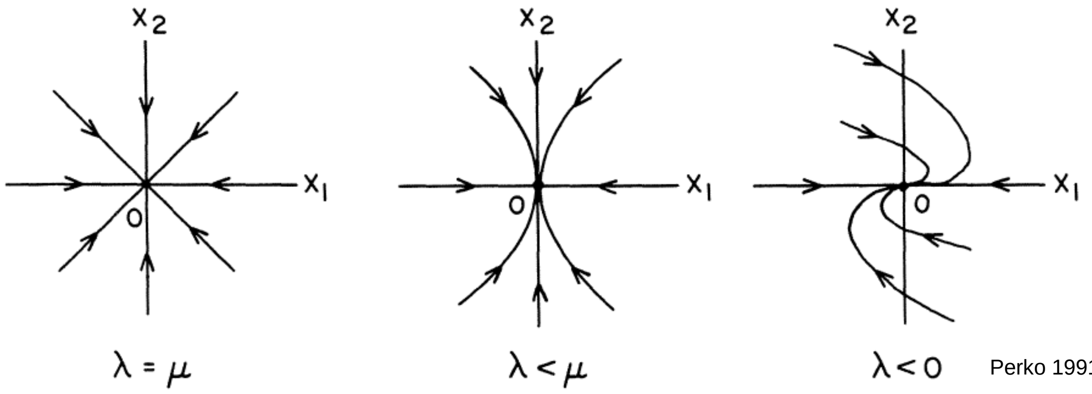


Perko 1991

Case II:

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \text{ with } \lambda \le \mu < 0 \text{ or } B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \text{ with } \lambda < 0$$

phase portrait

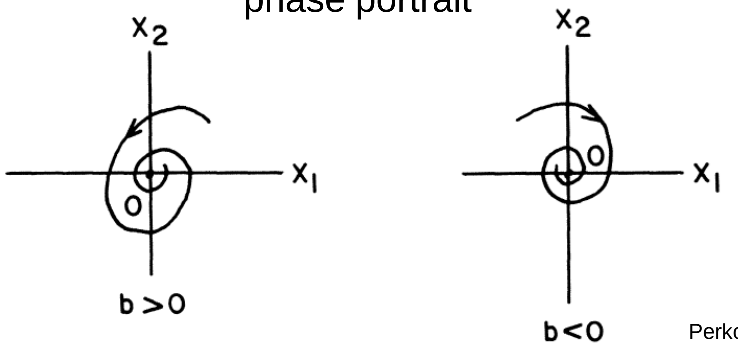


Perko 1991

Case III:

$$B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ with } a < 0$$

phase portrait



Perko 1991

2D Linear Systems: Phase Portraits

Linear Dynamical System

$$\dot{x} = Ax. \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Linear Dynamic System

$$\dot{x}_1 = ax_1 + bx_2$$
$$\dot{x}_2 = cx_1 + dx_2$$

Characterization

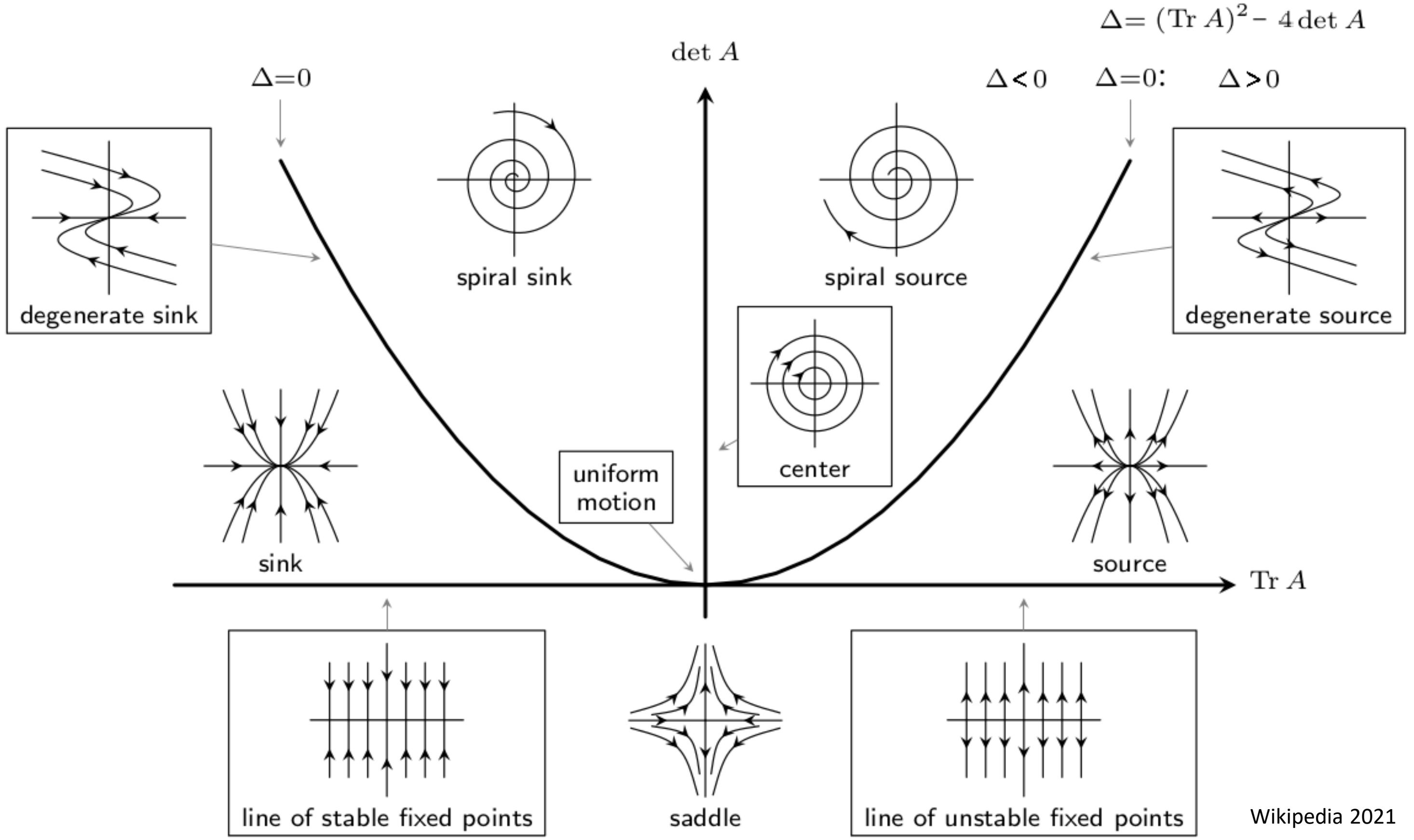
$$\det(A) = ad - bc$$

$$\mathrm{Tr}(A) = a + d$$

$$p(r) = r^2 - \mathrm{Tr}(A)\, r + \det(A).$$

$$r = \frac{\mathrm{Tr}(A) \pm \sqrt{(\mathrm{Tr}[A])^2 - 4\det[A]}}{2}$$

Linear System Behaviors in 2D



Wikipedia 2021

Conclusion

Enjoyed working with you all this quarter!

Thank You!