Finite Element Methods for Elasticity

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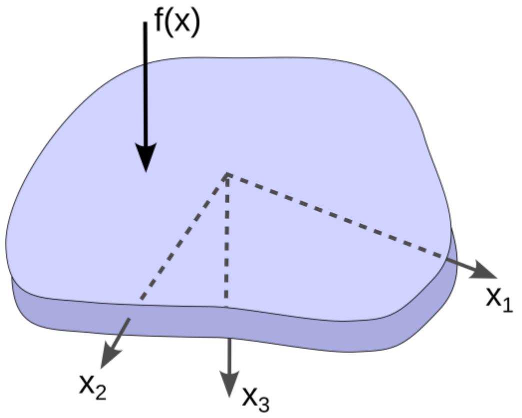
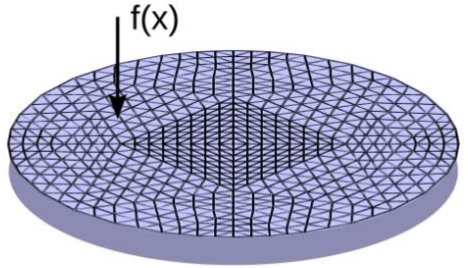
206D: Finite Element Methods
University of California Santa Barbara

Biharmonic PDE: Mechanics of plate bending

$$E_B \Delta^2 u = -f(x), \quad \mathbf{x} \in \Omega$$
$$\mathbf{n} \cdot \nabla u = 0, \quad \mathbf{x} \in \partial\Omega$$
$$u = 0, \quad \mathbf{x} \in \partial\Omega.$$

- $u(\mathbf{x})$ deflection in the z-direction.
- $f(\mathbf{x})$ load force in the z-direction.
- E_B bending modulus of the plate.



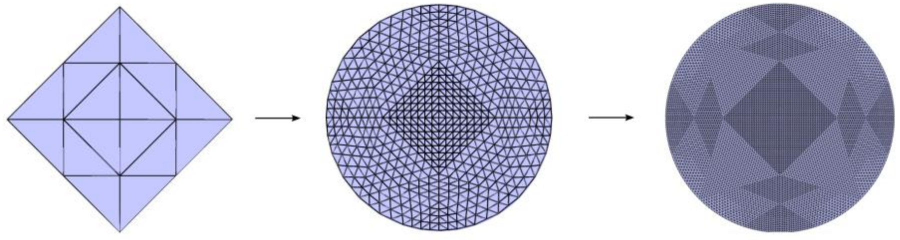


Finite Element Methods:

Fourth-order PDE $\rightarrow$ weak formulation has two derivatives.

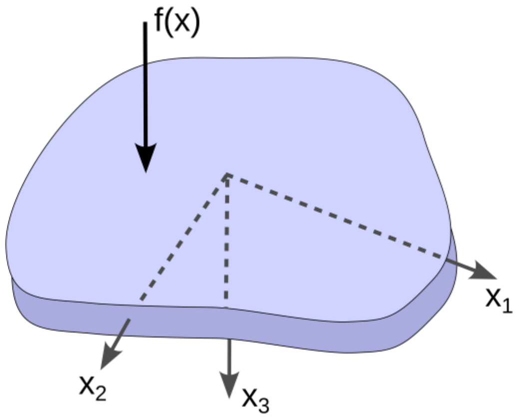
Conforming elements suggests we need C^1-regularity.

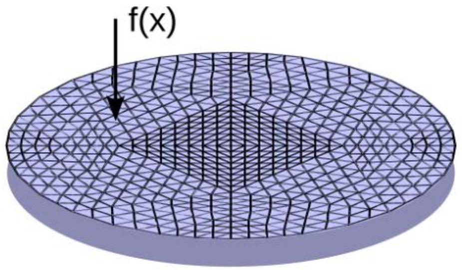
Developing effective C^1-elements poses challenges.



Elements: Hermite Cubic and Hermite Quintic

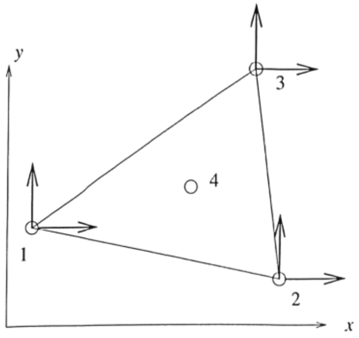
Finite Element Approximation



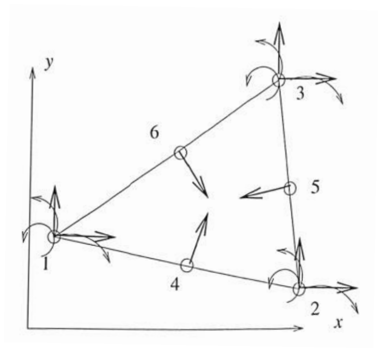


Candidate Elements

Hermite Cubic C0
(4 nodes , 10 DOF)



Hermite Quintic C1
(6 nodes , 21 DOF)



Hermite Cubic

Continuous first derivatives at nodes, but NOT along edges!

Non-conforming first derivatives along edges, only → C0.

Poor accuracy in practice for elasticity problems.

Hermite Quintic

Uses first and second derivatives at nodes.

Uses normal derivatives at midpoints.

Conforming first derivatives along edges → C1.

Considerations

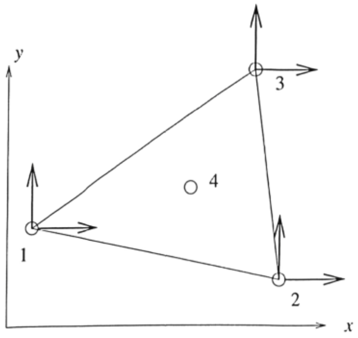
Hermite Quintic elements yield accurate approximations for elasticity problems.

However, expensive with 21 DOF.

Can accurate elements be developed with fewer DOF?

Candidate Elements

Hermite Cubic C0
(4 nodes , 10 DOF)



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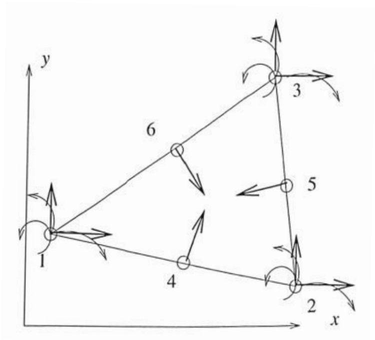
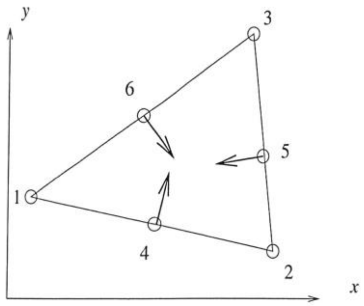
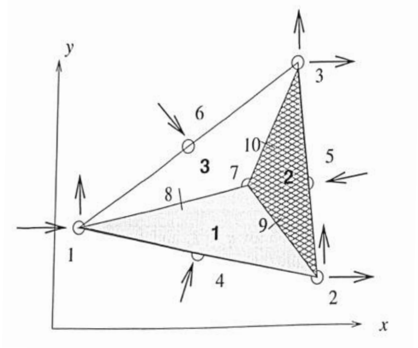


Morley Quadratic C0
(6 nodes , 6 DOF)



Hsieh-Clough-Tocher C1
(7 nodes , 12 DOF, macroelement 3-cubics)



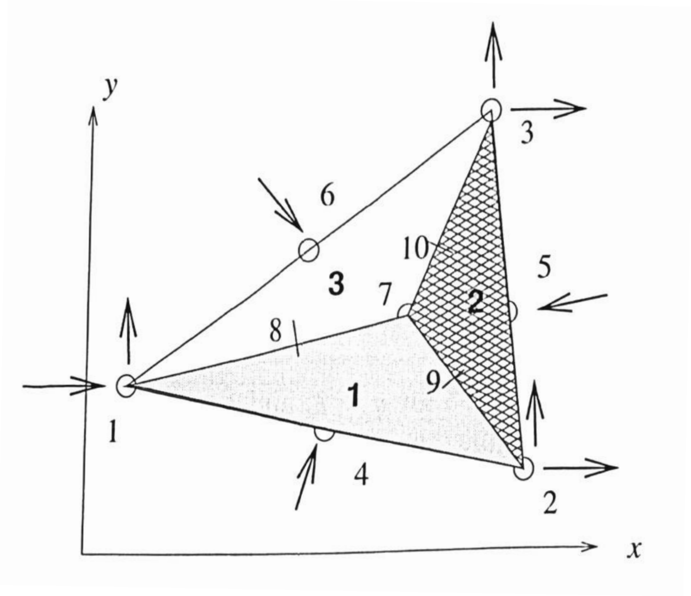
Morley Quadratic

Only 6 DOF → uses values at nodes and normal derivatives at midpoint edges.
However, non-conforming → C0.
Still yields accurate results for many elasticity problems.

Hsieh-Clough-Tocher (HCT)

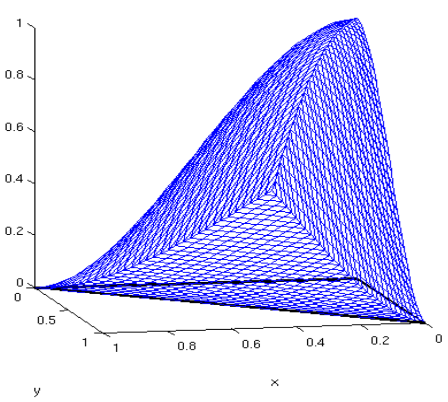
Macroelement divided into three parts with each using a cubic.
Cubics on each part coupled with C1 continuity imposed along interior edges.
Uses first derivatives at triangle vertices and normal derivatives at edge midpoints.

Conforming → C1 → well-founded convergence theory.
12 DOF → provides good trade-off for many elasticity problems.

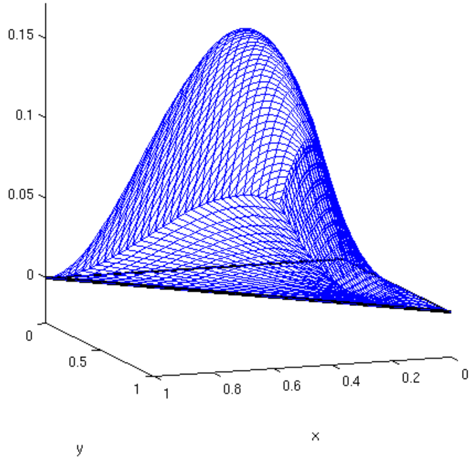


Hsieh-Clough-Tocher (HCT): Nodal Basis Functions

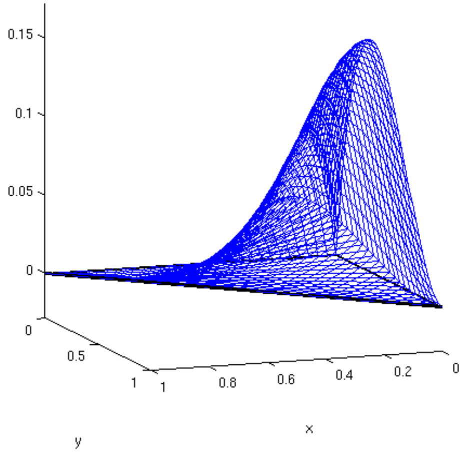
Node 1



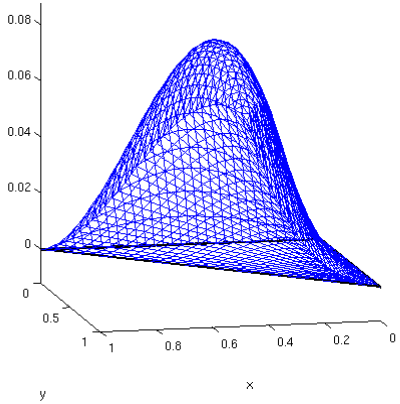
Node 2



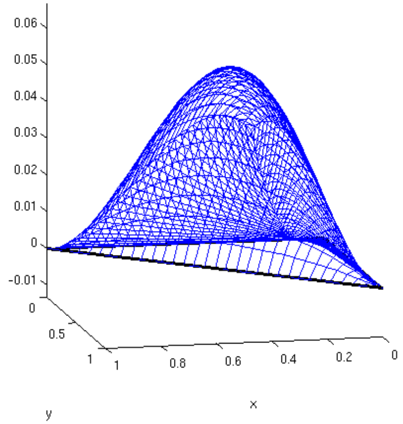
Node 3



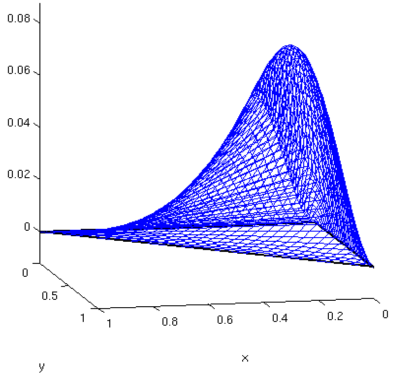
Node 10



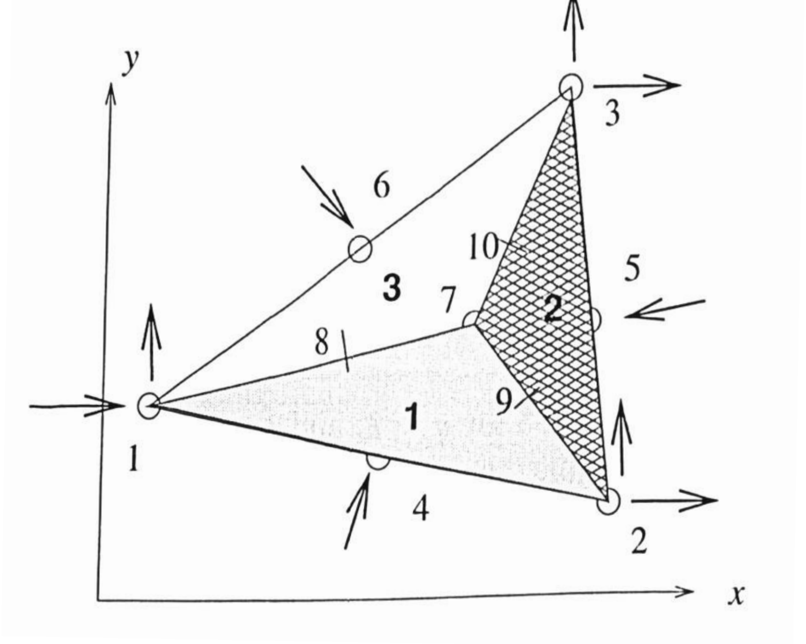
Node 11



Node 12



HCT Element



(see movies)

Remarks

Nodes 1-3, 4-9 similar to the hermite elements.
Nodes 10-12 similar to bubble nodes.

Cubics facilitate quadratures using standard methods over parts.
HCT is widely-used element for elasticity.

Biharmonic PDE: Mechanics of plate bending

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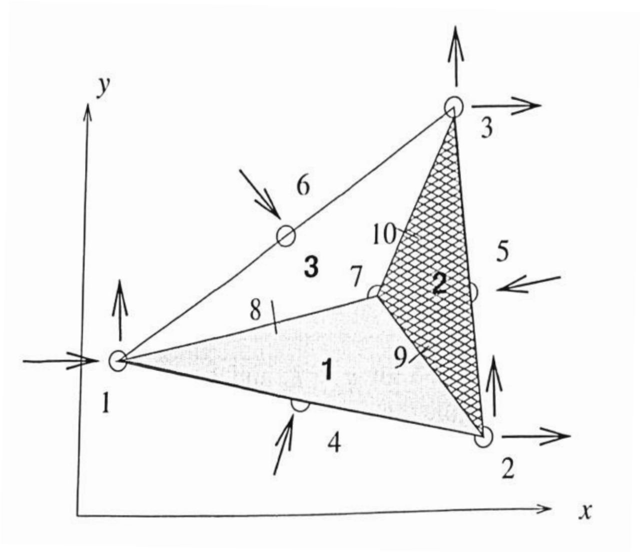
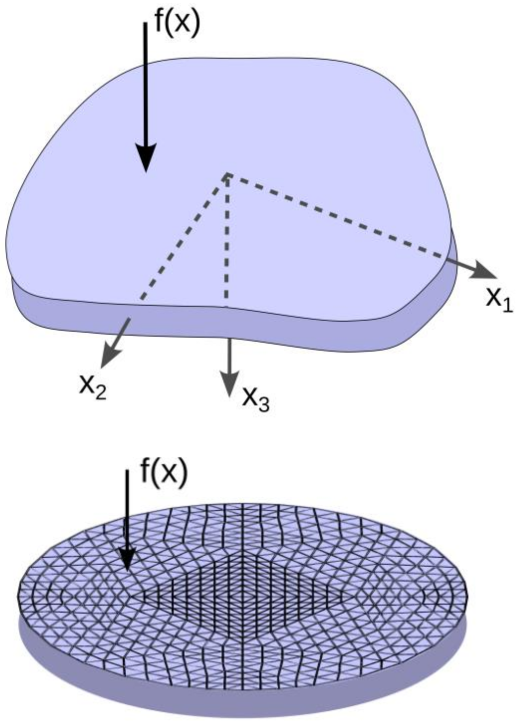




Numerical Solution

(i) variational formulation, (ii) meshing, (iii) assembly, (iv) linear solver.

HCT Elements → Ritz-Galerkin Approximation.

Example

Consider case with $f(x) = 1, EB = 1$ on disk.

By rotational symmetry becomes PDE

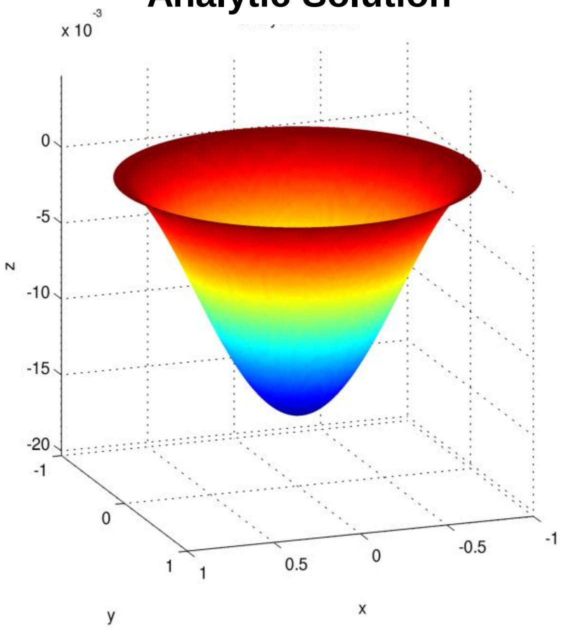
$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right)\right) = -f(x) \longrightarrow$$ quintic polynomial in r.

Numerical Results: Hsieh-Clough-Tocher (HCT) Elements

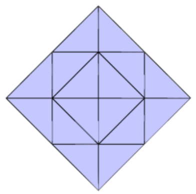
Biharmonic PDE:

$$E_B \Delta^2 u = -f(x), \quad \mathbf{x} \in \Omega$$
$$\mathbf{n} \cdot \nabla u = 0, \quad \mathbf{x} \in \partial\Omega$$
$$u = 0, \quad \mathbf{x} \in \partial\Omega.$$
$$f(x) = 1, E_B = 1$$

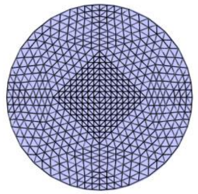
Analytic Solution



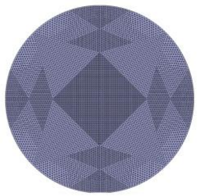
Level 0

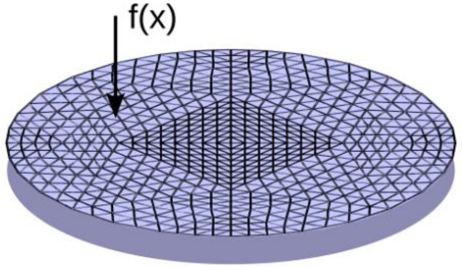


Level 3

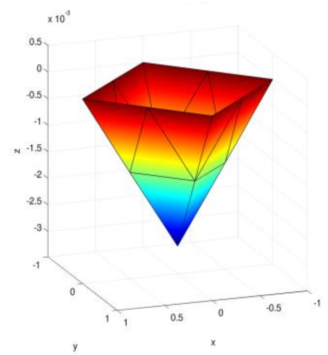


Level 5





Solution



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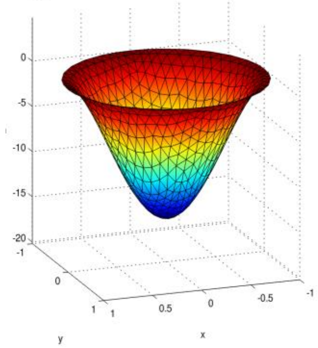

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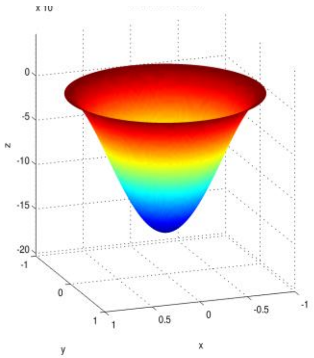
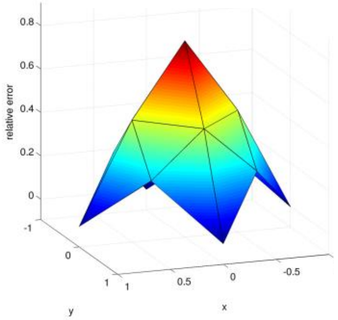
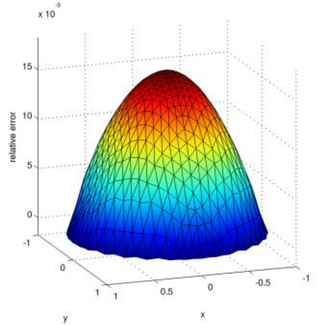


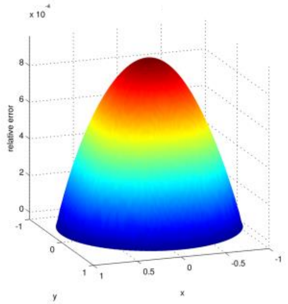
Error: 8.1e-1



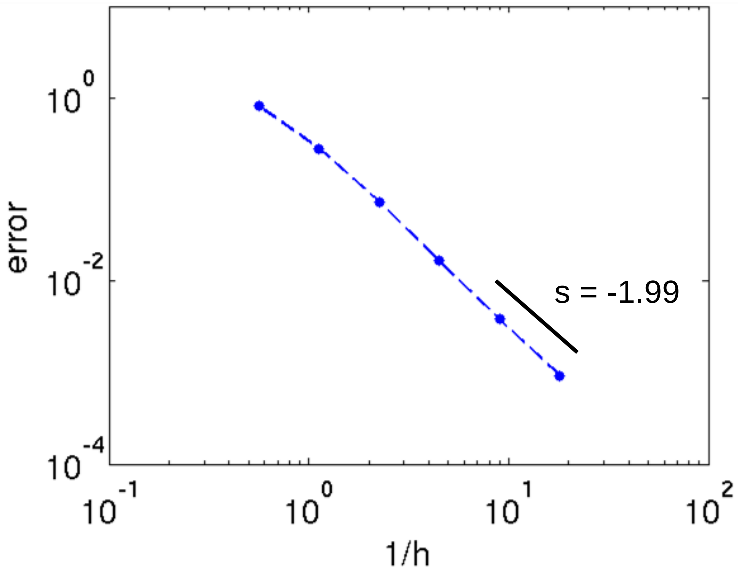
Error: 1.7e-2



Error: 9.2e-04



Convergence



HCT elements converge with second-order accuracy.