# FEM Approximation Properties and Convergence

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206D: Finite Element Methods
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# Approximation by Finite Elements

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**Goal:** Obtain estimates of $\|v - I_h v\|_{m,h}$ in terms of $\|v\|_{t,\Omega}$ and $h$ with $m \leq t$.

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For a bounded domain $\Omega$, the **chunkiness parameter** $\gamma$ is defined to be the ratio of the diameter $d_\Omega$ of $\Omega$ to the largest radius $r_{max}$ for which $\Omega$ is star-shaped, $\gamma = d_\Omega / r_{max}$.

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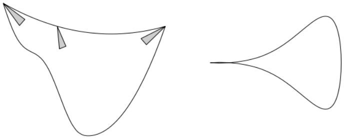

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An open domain $\Omega$ is said to satisfy the **cone condition** with angle $\phi$ and radius $r$ if at every point $\mathrm{x} \in \Omega$ we have $\mathrm{x} + \mathcal{C}_{\phi,r,\mathrm{e}_\mathrm{x}} \subset \Omega$ for some orientation $\mathrm{e}_\mathrm{x}$.

## Lemma

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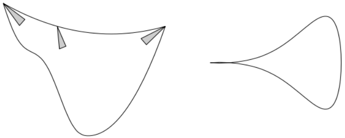

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## Lemma

Consider an $\Omega$ that is bounded and star-shaped with respect to $\mathcal{B}(\mathrm{x}_c, r_c)$ and contained within $\mathcal{B}(\mathrm{x}_c, R)$. Then $\Omega$ satisfies an **interior cone condition** with radius $r_c$ and angle $\phi = 2\arcsin\left(r_c/2R\right)$.

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$$\|u - \mathcal{I}_s u\|_t \leq c|u|_t, \quad \forall u \in H^t(\Omega).$$

**Proof:** Let

$$|||v||| := |v|_t + \sum_{i=1}^{s} |v(z_i)|.$$

We show the norms $|||\cdot|||$ and $\|\cdot\|_t$ are equivalent. If this were the case, the bound would follow from

$$\|u - \mathcal{I}_s u\|_t \leq c|||u - I_s u||| = c\left(|u - \mathcal{I}_s u|_t + \sum_{i=1}^{s} |(u - \mathcal{I}_s u)(z_i)|\right) = c|u - \mathcal{I}_s u|_t = c|u|_t.$$

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Let $\Omega \subset \mathbb{R}^2$ be domain with Lipschitz continuous boundary. Suppose $t \geq 2$ and $L$ is a bounded linear mapping of $H^t(\Omega)$ into a normed linear space $\mathcal{Z}$. If $\mathcal{P}_{t-1} \subset \ker(L)$, then there exists a constant $c = c(\Omega)\|L\| \geq 0$, so that

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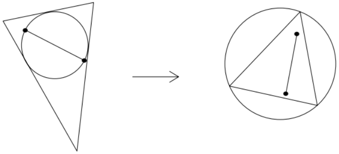

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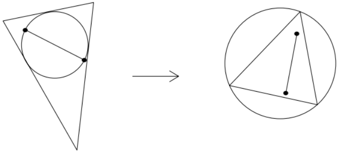

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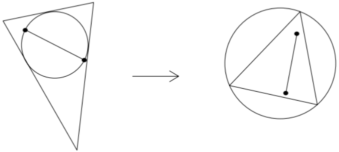

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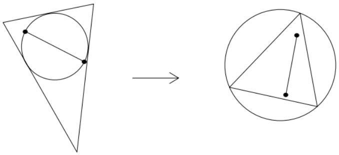

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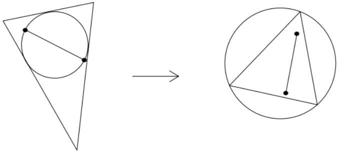

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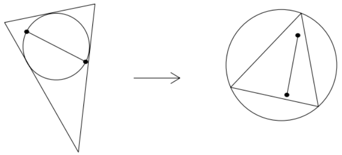

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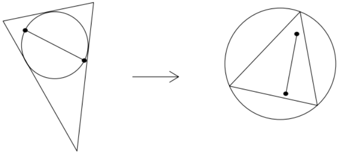

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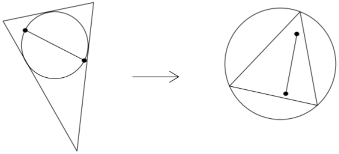

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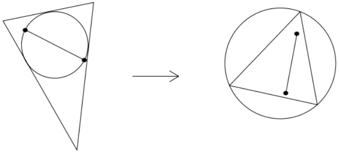

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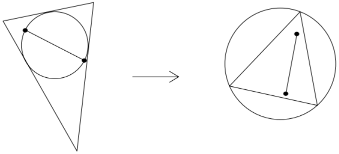

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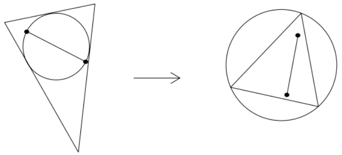

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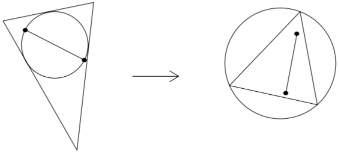

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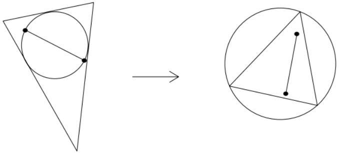

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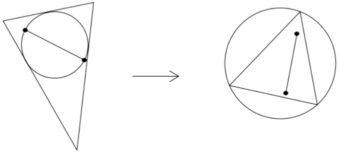

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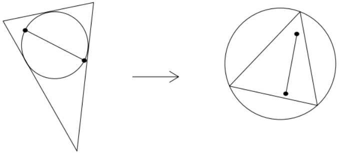

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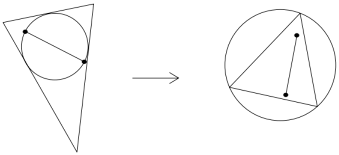

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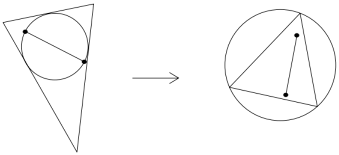

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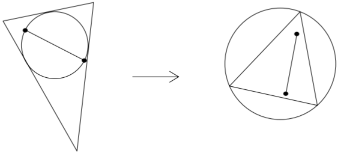

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# Approximation by Finite Elements

## Theorem for Quadrilateral Bilinear Elements

Consider $\mathcal{T}_h$ a quasi-uniform decomposition of $\Omega$ into parallelograms. There exists a constant $c = c(\Omega, \kappa)$ such that

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$$\|u - \mathcal{I}_h u\|_2 \leq \|u\|_2 + \|\mathcal{I}_h u\|_2 \leq (c_2 + 1)\|u\|_2.$$

When $u$ is linear polynomial then $\mathcal{I}_h u = u$ and $u - \mathcal{I}_h u = 0$.
By Bramble-Hilbert II we have the result. ■

**Remark:** For Serendipity Elements a similar proof technique can be used to obtain $\|u - \mathcal{I}_h u\|_{m,\Omega} \leq ch^{t-m}|u|_{t,\Omega}, \ \forall u \in H^t(\Omega), \ m = 0, 1, \ t = 2, 3.$

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Consider affine family of elements $\{\mathcal{S}_h\}$ with piecewise polynomials of degree $k$ having uniform partitions. There exists a constant $c = c(\kappa, k, t)$ so that

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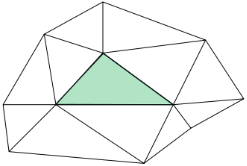

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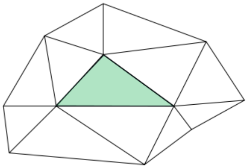

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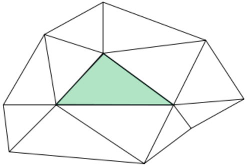

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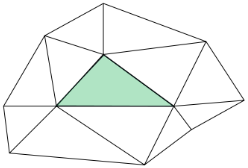

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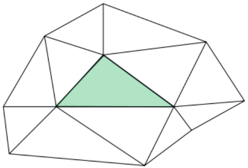

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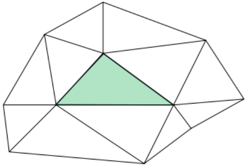

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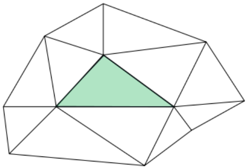

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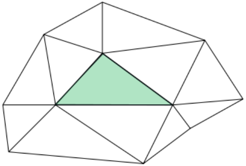

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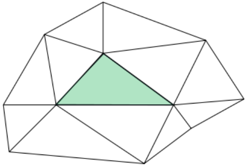

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How do we construct such an operator $\mathcal{I}_h$ in practice?

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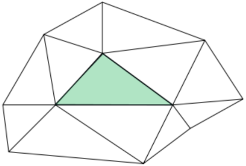

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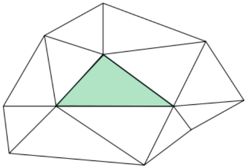

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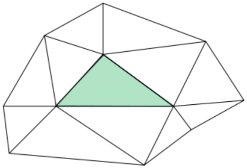

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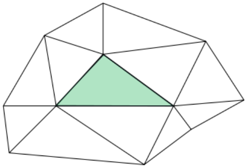
$$\bar{Q}_j v = \begin{cases} 0 & \text{if } x_j \in \Gamma_D \\ Q_j v & \text{otherwise.} \end{cases}, \qquad Q_j : L_2(\omega_j) \to \mathcal{P}_0.$$



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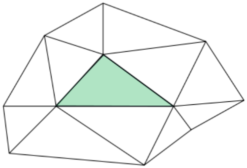

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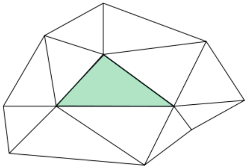

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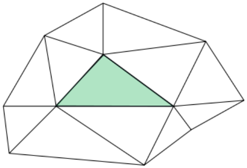

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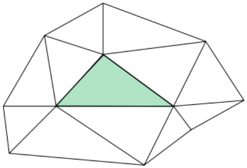

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For a given nodal point $x_j$ let

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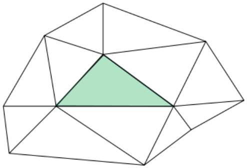

The $Q_j$ is the $L_2$-projection onto constant functions.
The $\Gamma_D \subset \partial\Omega$ is part with Dirichlet boundary conditions.

**Clément's Interpolation:**

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The cardinal shape functions $v_j$ form a partition of unity for elements (one on node $j$, zero at other nodes).

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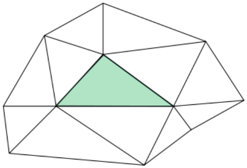

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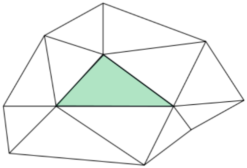

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