Finite Element Spaces

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206D: Finite Element Methods
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and $L \equiv 0$ implies $\alpha_i = 0$.

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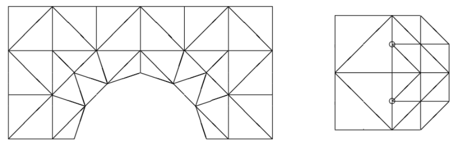

admissible triangulation

inadmissible (hanging nodes)

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For a bounded domain Ω, admissible partition, and $k \geq 1$, a piecewise infinitely differentiable function $v : \overline{\Omega} \to \mathbb{R}$ belongs to $H^k(\Omega)$ if and only if $v \in C^{k-1}(\overline{\Omega})$.

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Other shape spaces, partition types, and non-conforming finite elements are also possible.

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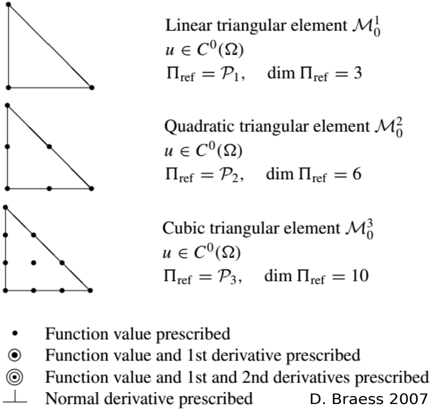

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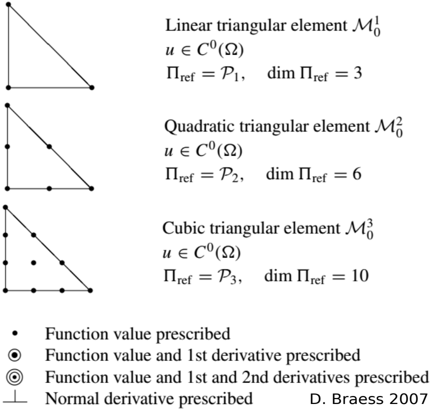

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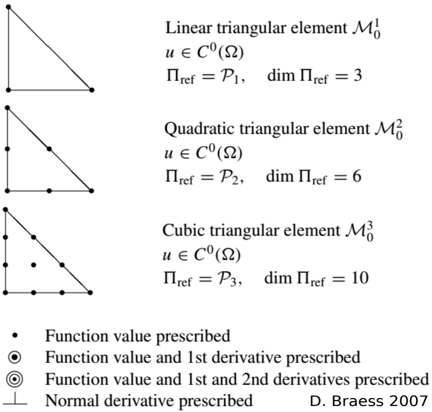

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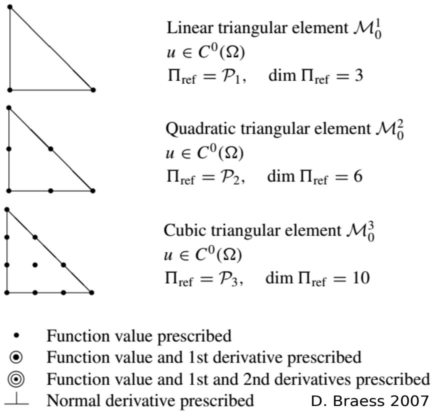

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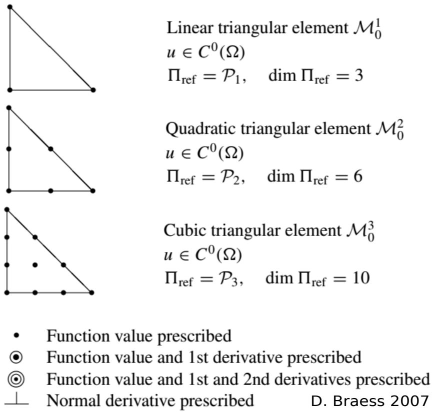

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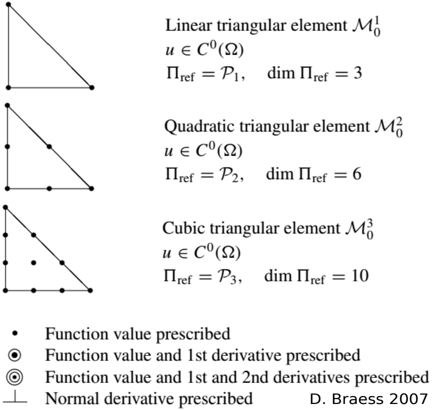

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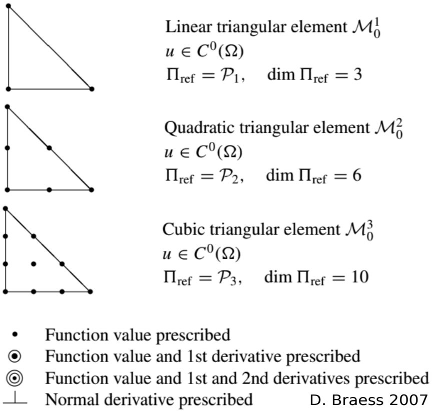

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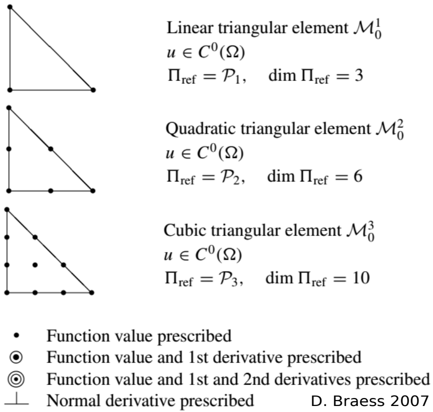

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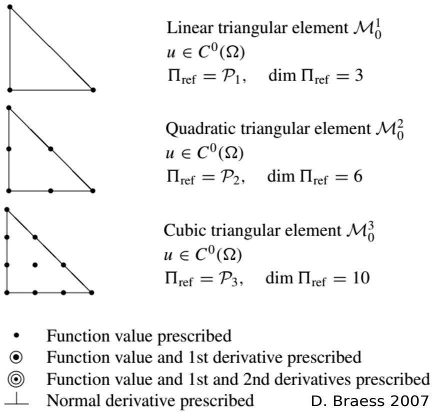

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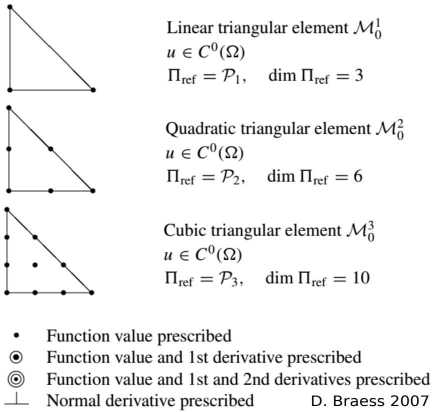

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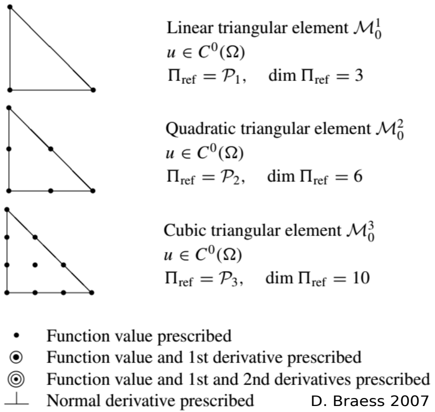

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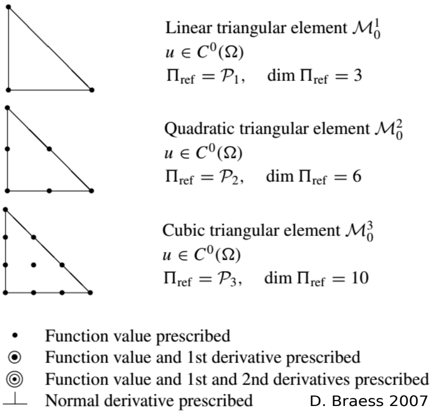

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Triangular Finite Elements: Lagrange Elements

Consider partition of the domain into triangular elements $\mathcal{T}$.

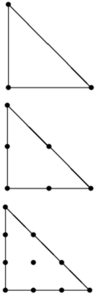
Lemma

Consider triangle T with $z_1, z_2, \ldots, z_s$, $s = 1 + 2 + \cdots (t + 1)$ nodes lying on the lines depicted. For every $f \in C(T)$ there is a unique polynomial p of degree $\leq t$ satisfying interpolation

$$p(z_i) = f(z_i), \quad 1 \leq i \leq s.$$

Proof: We proceed by induction. Clearly, in the case of $t = 0$ when $s = 1$ we have interpolation by the constant polynomials.

Now if the interpolation for $t - 1$ holds, we prove it holds for t. Let p_1 be the univariate Lagrange polynomial interpolating the $t + 1$ points on the x-axis. Consider the sub-triangle neglecting the points on the x-axis. Let p_2 be the interpolating polynomial for these points with $p_2(z_i) = (f(z_i) - p_1(z_i))/y_i$, $1 \leq i \leq s - (t + 1)$. The polynomial $q(x, y) = p_1(x) + y p_2(x, y)$ interpolates all points. Uniqueness as exercise (use holds for degree $t - 1$). ■



Linear triangular element $\mathcal{M}_0^1$
$u \in C^0(\Omega)$
$\Pi_{\text{ref}} = \mathcal{P}_1, \quad \dim \Pi_{\text{ref}} = 3$

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$u \in C^0(\Omega)$
$\Pi_{\text{ref}} = \mathcal{P}_2, \quad \dim \Pi_{\text{ref}} = 6$

Cubic triangular element $\mathcal{M}_0^3$
$u \in C^0(\Omega)$
$\Pi_{\text{ref}} = \mathcal{P}_3, \quad \dim \Pi_{\text{ref}} = 10$

- • Function value prescribed
- ⊙ Function value and 1st derivative prescribed
- ◎ Function value and 1st and 2nd derivatives prescribed
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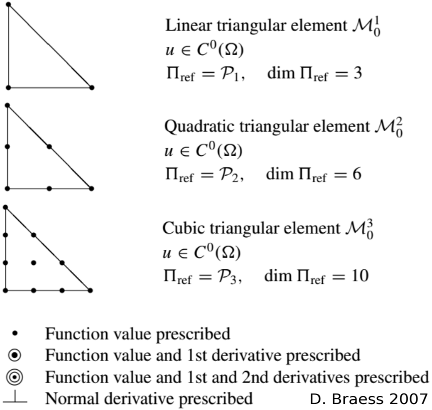
Triangular Finite Elements: Lagrange Elements

Consider partition of the domain into triangular elements $\mathcal{T}$.



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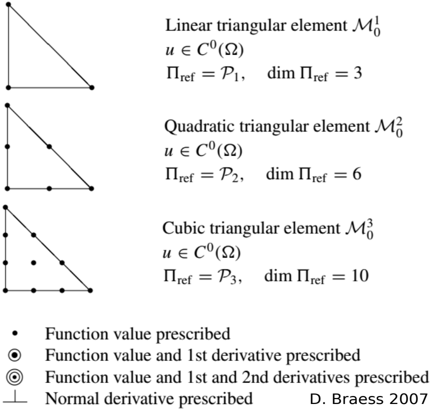

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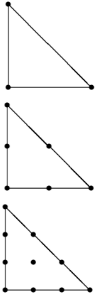
Triangular Finite Elements: Lagrange Elements

Consider partition of the domain into triangular elements $\mathcal{T}$.

Definition

$$\mathcal{M}^k := \mathcal{M}_k(\mathcal{T}) := \{v \in L^2(\Omega);\ v|_{\mathcal{T}} \in \mathcal{P}_t \text{ for every } T \in \mathcal{T}\}$$



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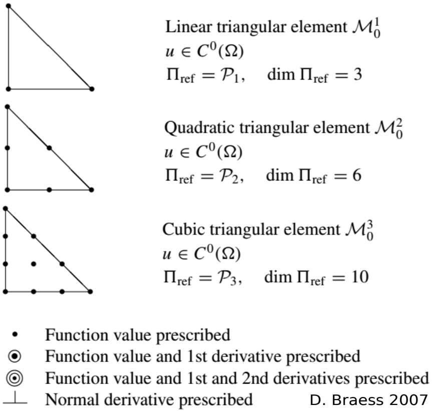

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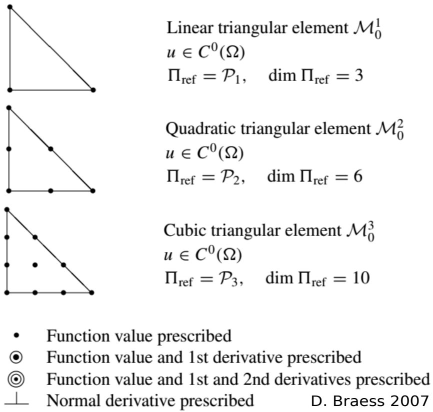

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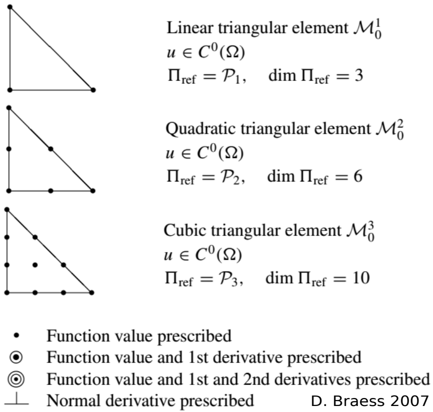
The $\mathcal{M}_0^k$ provide C^0 elements $\subset H^1$.



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Triangular Finite Elements: Lagrange Elements

Consider partition of the domain into triangular elements $\mathcal{T}$.

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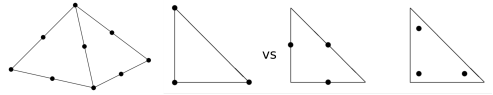

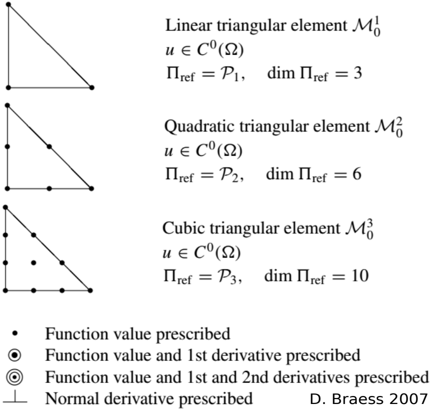

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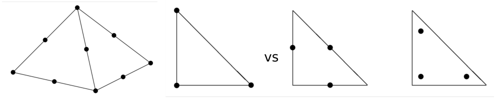
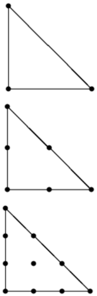


Note: Shared common nodes at vertices ensures the continuity.



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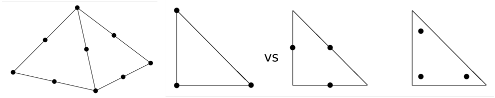

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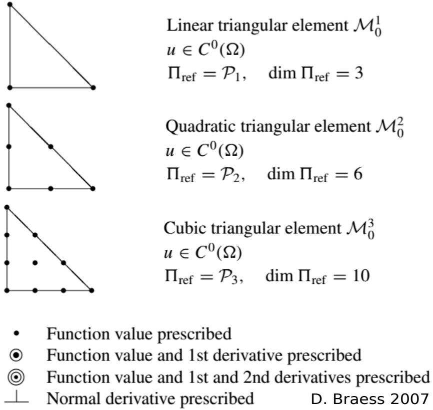
$\mathcal{M}_0^k$ is called the **conforming** P_k **element**.



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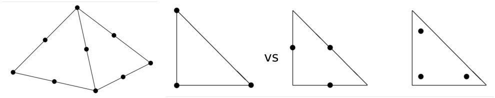

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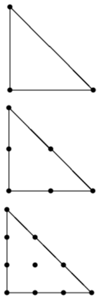
$\mathcal{M}_0^k$ is called the **conforming** P_k **element**.
$\mathcal{M}_0^1$ is sometimes called the **Courant triangle**.



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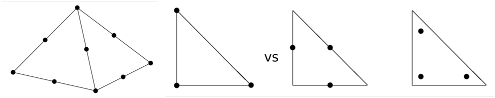

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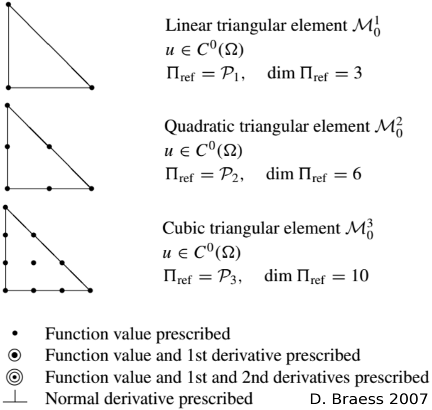
Nodal variables are $N_j(u) = u(z_j)$, so also called **Lagrange elements**.



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Triangular Finite Elements: C^1 Regularity

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More challenging to obtain elements with C^1 regularity.

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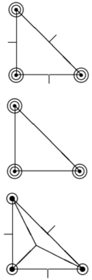

Argyris triangle
$u \in C^1(\Omega)$
$\Pi_{\text{ref}} = \mathcal{P}_5, \quad \dim \Pi_{\text{ref}} = 21$

Bell triangle
$u \in C^1(\Omega)$
$\Pi_{\text{ref}} \subset \mathcal{P}_5,\ \partial_\nu u|_{\partial T_i} \in \mathcal{P}_3, \quad \dim \Pi_{\text{ref}} = 18$

Hsieh–Clough–Tocher element
$u \in C^1(\Omega)$
$T = \bigcup_{i=1}^{3} K_i, \quad u|_{K_i} \in \mathcal{P}_3, \quad \dim \Pi_{\text{ref}} = 12$

- • Function value prescribed
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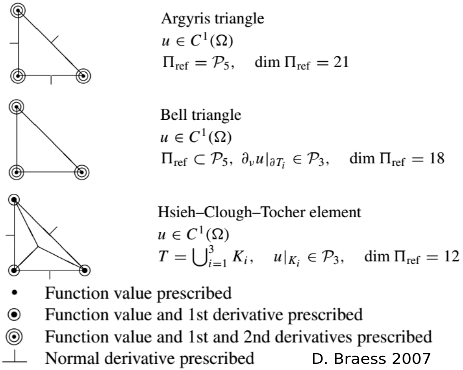

Triangular Finite Elements: C^1 Regularity

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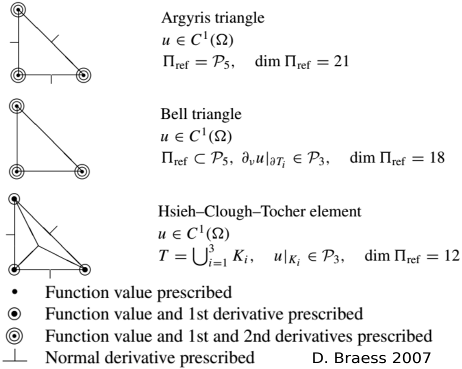
Argyris element:
Uses $\mathcal{P}_5$ which has $\dim \mathcal{P}_5 = 21$.



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Argyris element:
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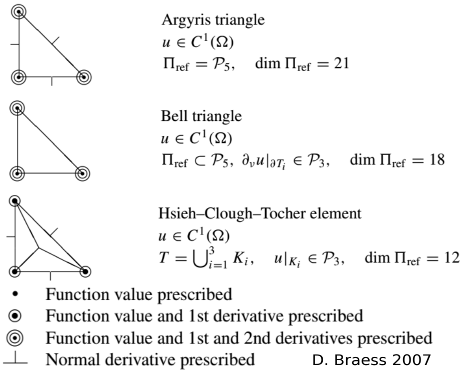
Values given of all derivatives up to order 2 at the vertices.



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Argyris element:
Uses $\mathcal{P}_5$ which has $\dim \mathcal{P}_5 = 21$.
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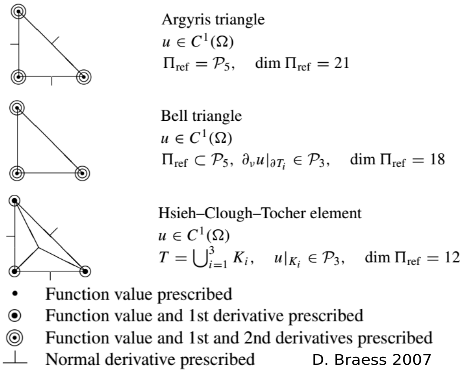
However, this is only $3 \times 6 = 18$ DOF.



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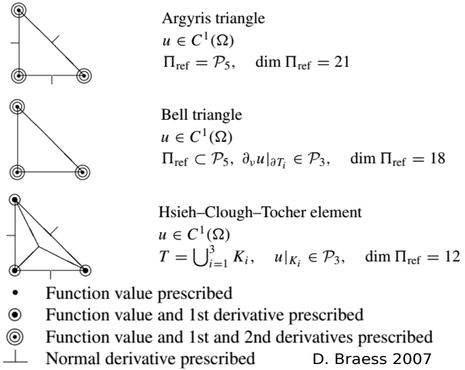
Determine 3 DOF from normal derivative at edge centers.



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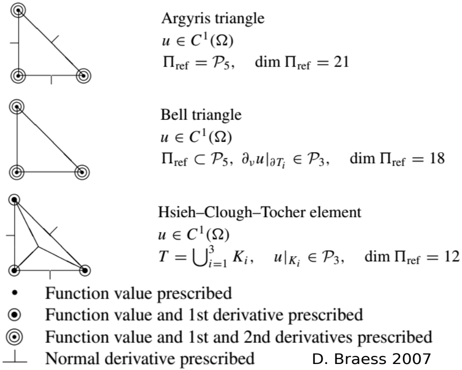

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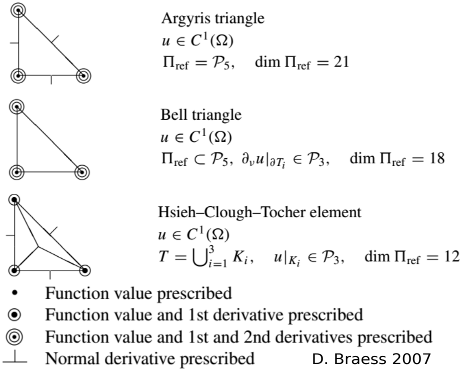

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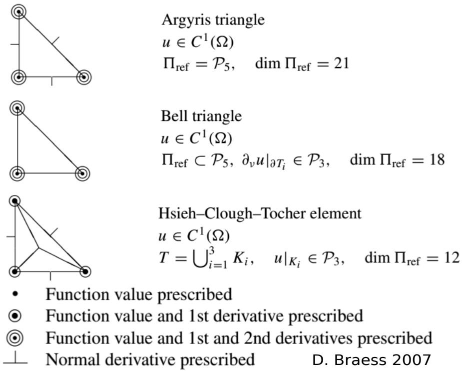
$\tilde{\mathcal{P}}_5$ restricted to polynomials having normal derivatives along the edges only of degree 4, $(\partial_n p(\mathbf{x}_e) \in \mathcal{P}_4)$).



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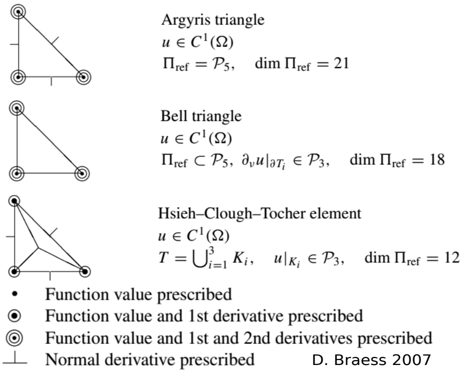

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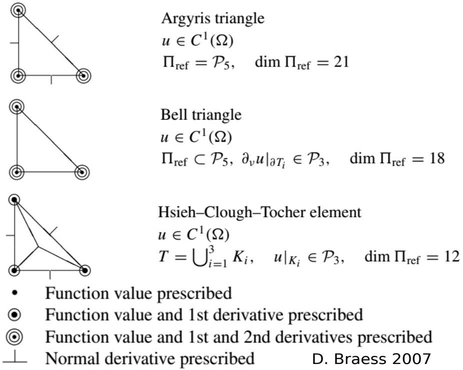
Hsieh-Clough-Tocher element:



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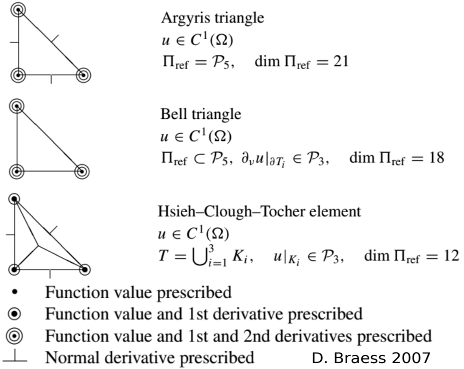
Hsieh-Clough-Tocher element: Macroelement approach.



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Values given of all derivatives up to order 2 at the vertices.
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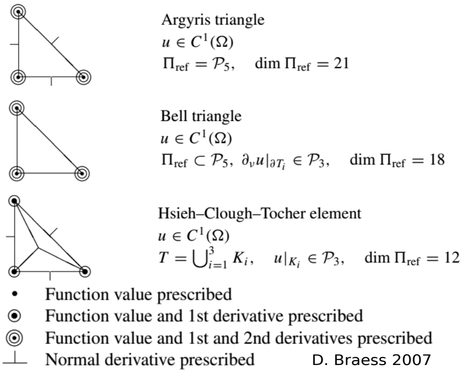

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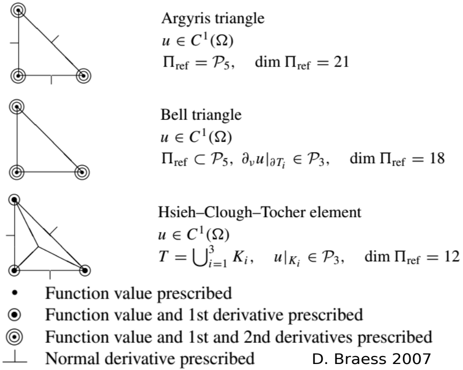

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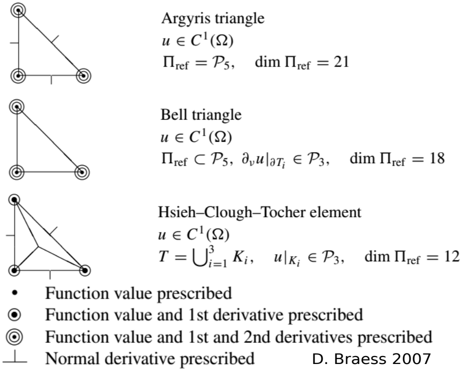

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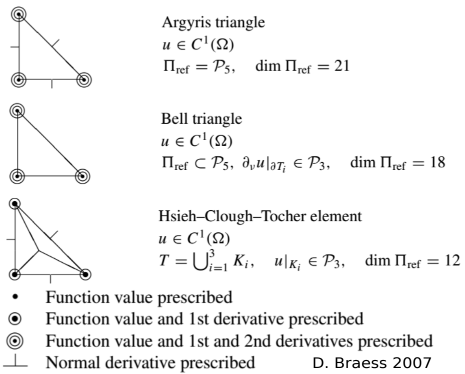

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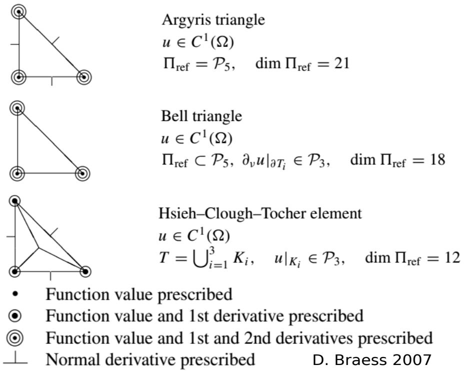

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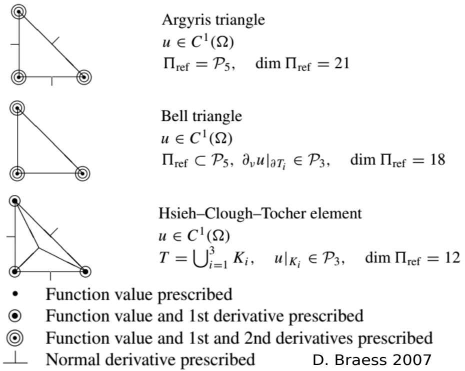

Bernstein-Bézier representation of polynomials used to handle derivatives along element boundaries.

Quadrilateral Finite Elements

Tensor Product Bases

Quadrilateral Finite Elements

Tensor Product Bases

A **tensor-product basis** generated by $\{\phi_k\}_{k=1}^{t}$ for $\mathbf{x} \in \mathbb{R}^n$

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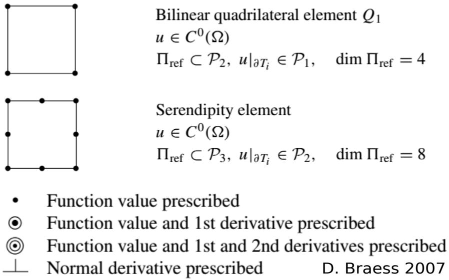

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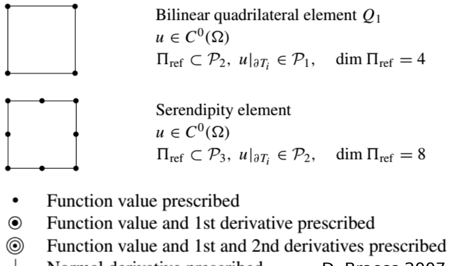

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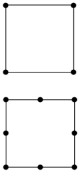

Bilinear quadrilateral element Q_1
$u \in C^0(\Omega)$
$\Pi_{\text{ref}} \subset \mathcal{P}_2, \ u|_{\partial T_i} \in \mathcal{P}_1, \quad \dim \Pi_{\text{ref}} = 4$

Serendipity element
$u \in C^0(\Omega)$
$\Pi_{\text{ref}} \subset \mathcal{P}_3, \ u|_{\partial T_i} \in \mathcal{P}_2, \quad \dim \Pi_{\text{ref}} = 8$

- • Function value prescribed
- ⊙ Function value and 1st derivative prescribed
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Quadrilateral Finite Elements

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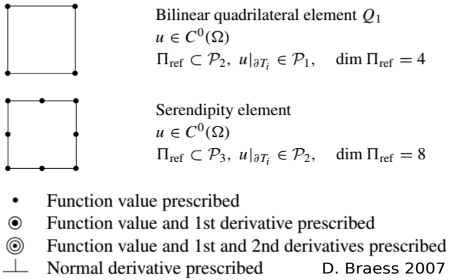

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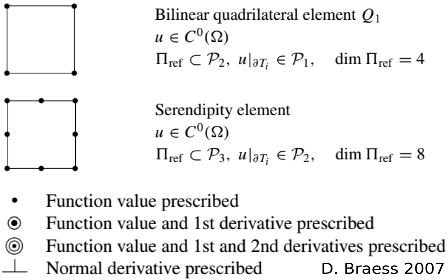
Quadrilateral Finite Elements



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Quadrilateral Finite Elements

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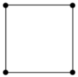

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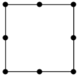

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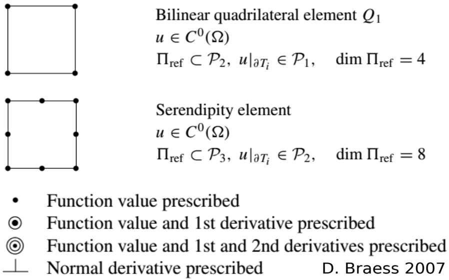

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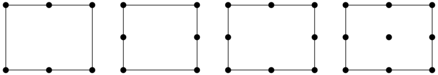

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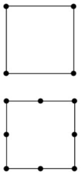

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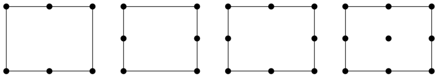

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p(x,y) = {} & c_0 + c_1 x + c_2 y + c_3 xy \\
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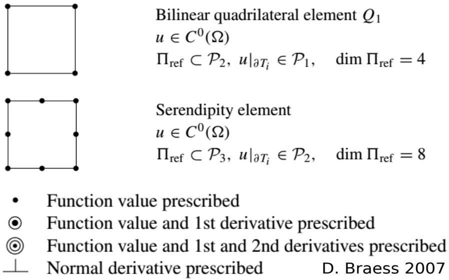
\end{aligned}
$$



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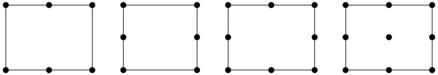

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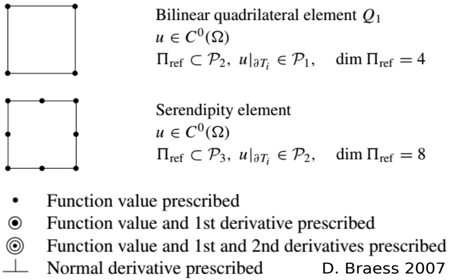
Nodal locations are vertices of rectangle and edge mid-points.



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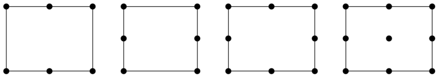

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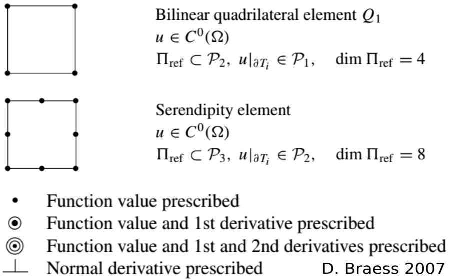

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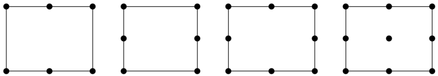

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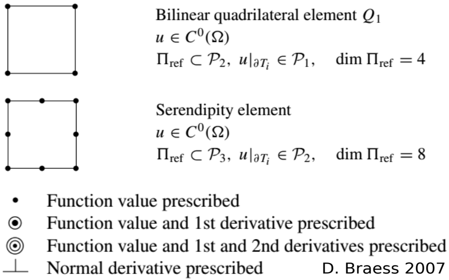
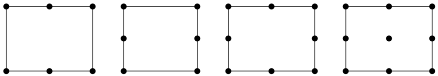
Consider $\mathcal{S}_9 = \mathcal{S}_{sd} \bigoplus \{c_8(x^2-1)(y^2-1)\}$.





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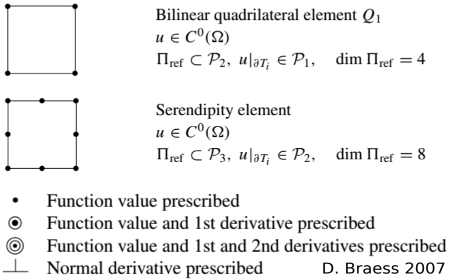

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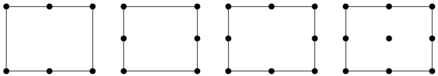

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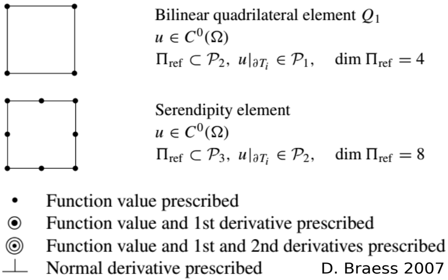
Add nodal location at the center of the rectangle.



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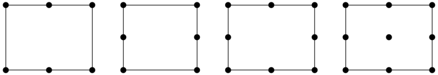

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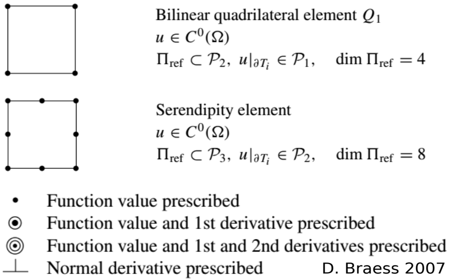

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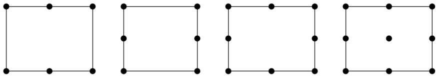

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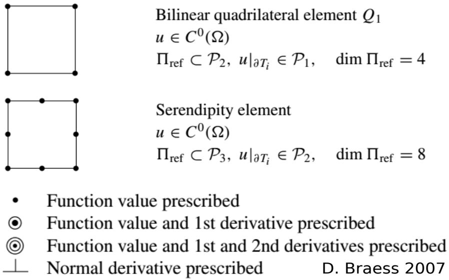

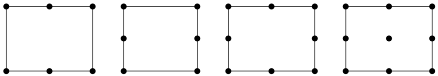

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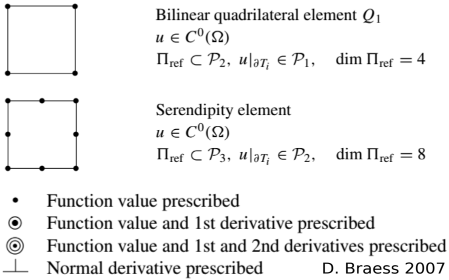

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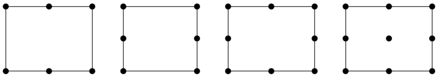

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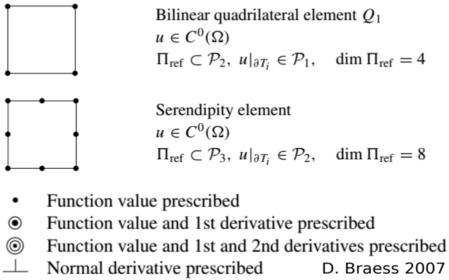

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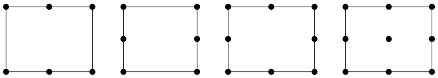

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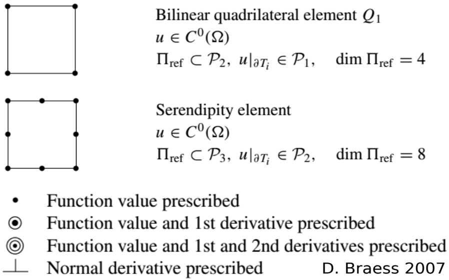

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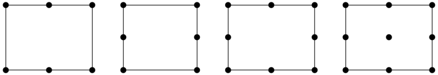

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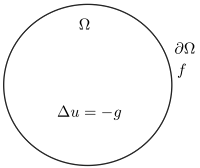

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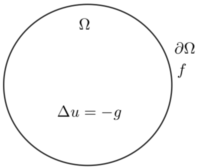

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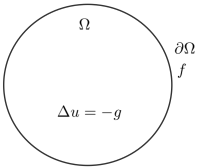

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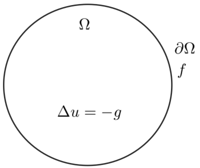

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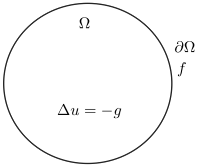

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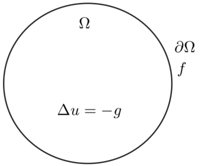

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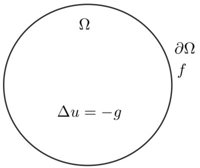

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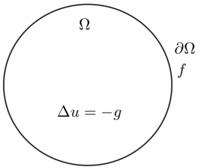

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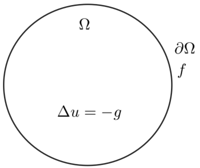

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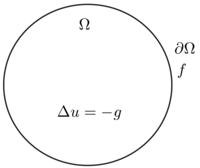

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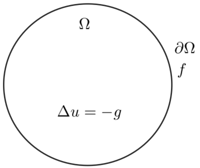

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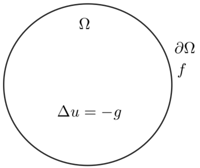

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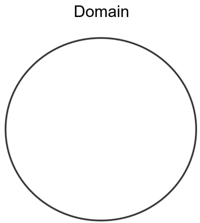
Divide domain into triangular elements T_j.



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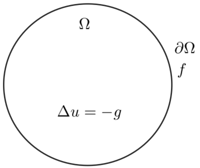
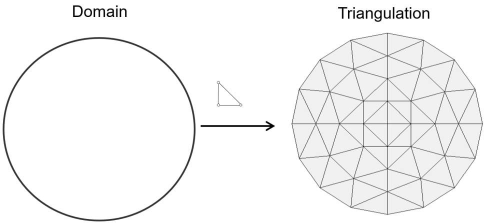
$$\left\{ \begin{array}{ll} \Delta u = -g, & x \in \Omega \\ u = f, & x \in \partial\Omega. \end{array} \right\} \rightarrow \left\{ \begin{array}{l} a(u,v) = -(g,v), \;\; v \in \mathcal{S} \\ a(u,v) = \int_\Omega \nabla u \cdot \nabla v d\mathbf{x}. \end{array} \right\} \text{ (RG-Approximation)}$$



Discretization:

Divide domain into triangular elements T_j.



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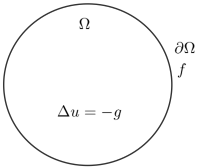

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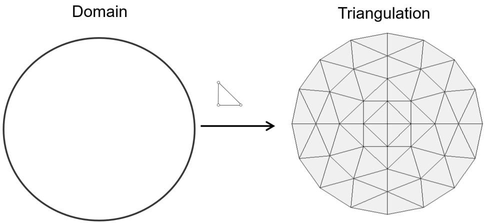

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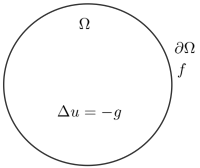

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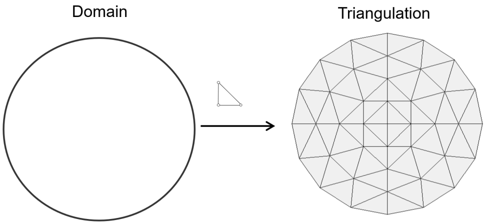

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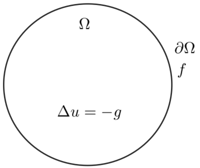

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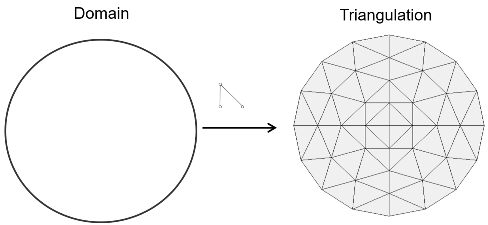

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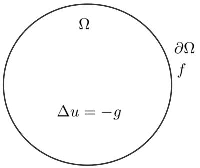

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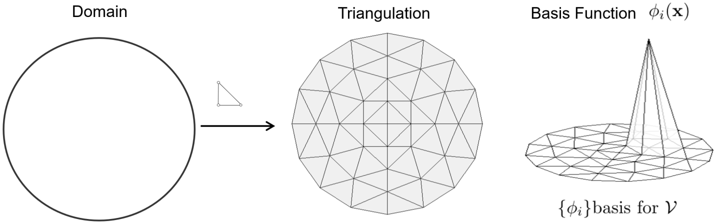

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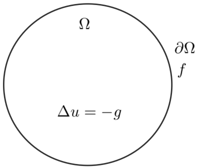

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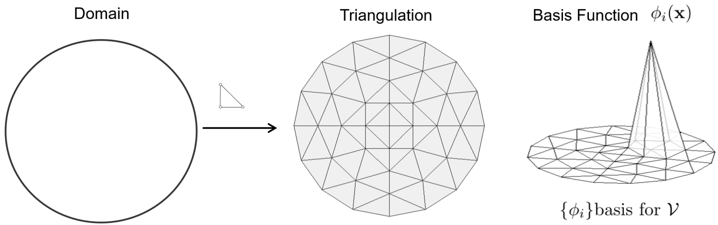

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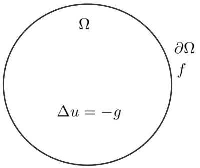

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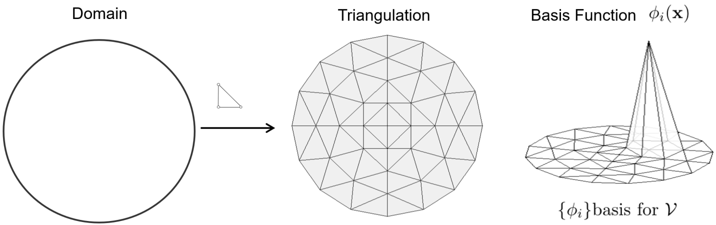
Functions in $v \in \mathcal{S}$ can be represented as

$$v(x) = \sum_{i=1}^{n} v(\mathbf{x}_i)\phi_i(\mathbf{x}) \in H^1.$$



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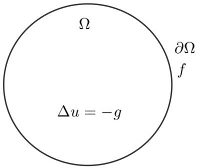

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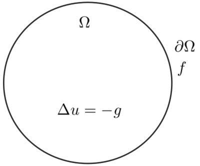

Mesh Refinement:

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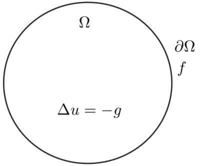

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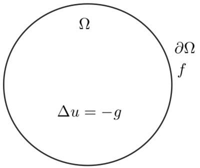

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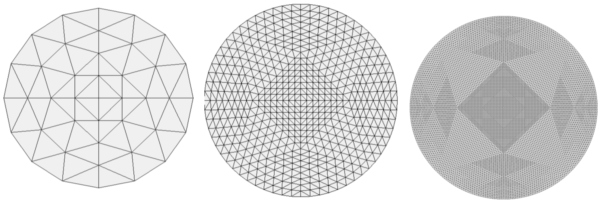

refinement = 2   refinement = 4   refinement = 6

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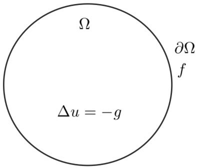

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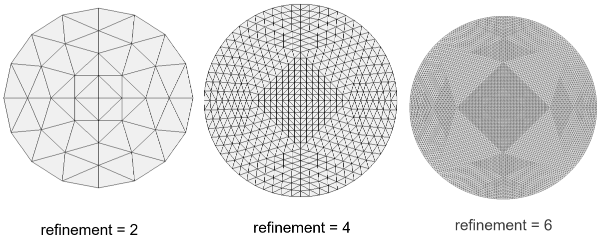

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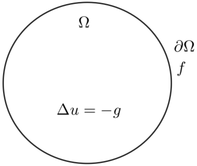

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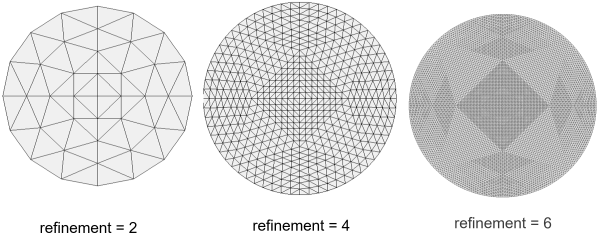

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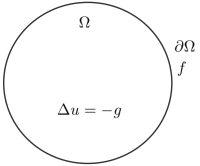

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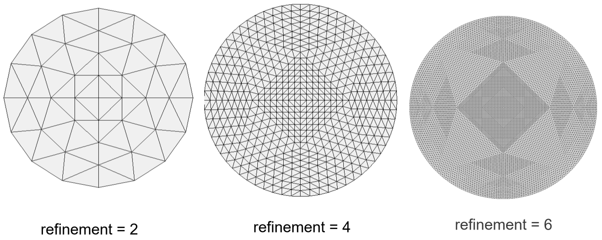

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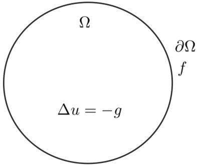

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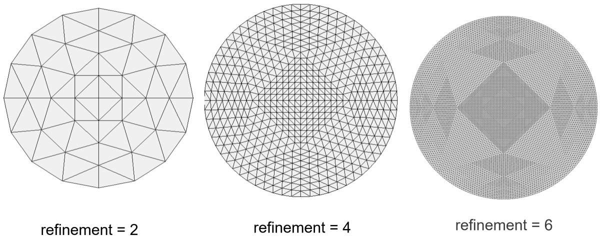

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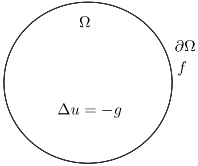

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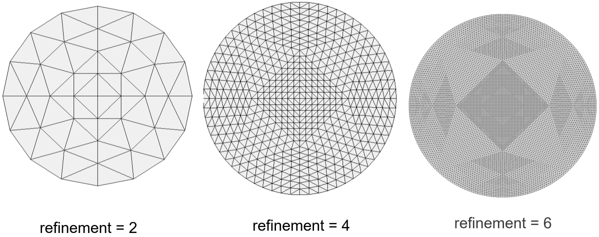

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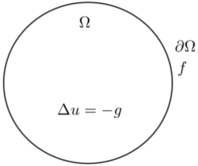

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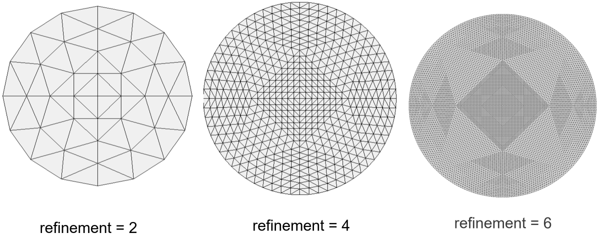
Convergence expected sufficiently uniform refinements.



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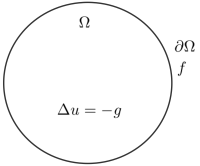

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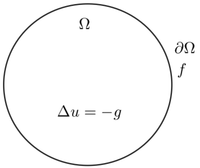

Example:

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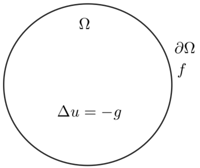

Example:

Consider PDE with

$$g(x,y) = \pi^2 \sin(\pi x) + \pi^2 \cos(\pi x)$$
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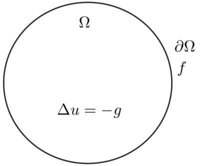

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$$f(x,y) = \sin(\pi x) + \cos(\pi x).$$

Solution is

$$u(x,y) = \sin(\pi x) + \cos(\pi x).$$

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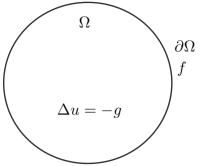

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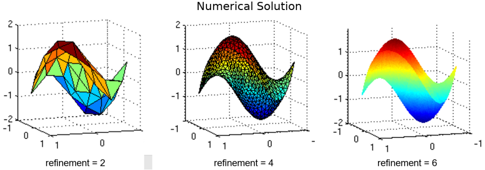

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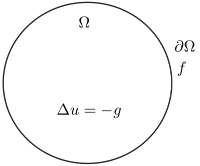

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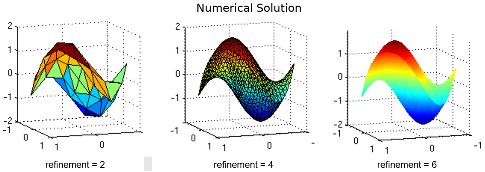

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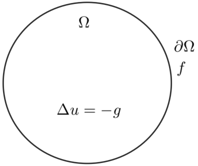

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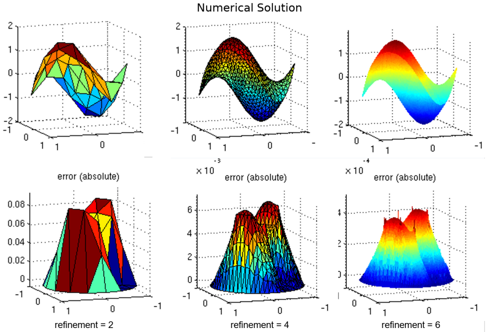

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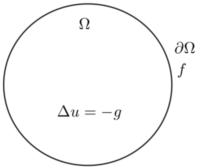

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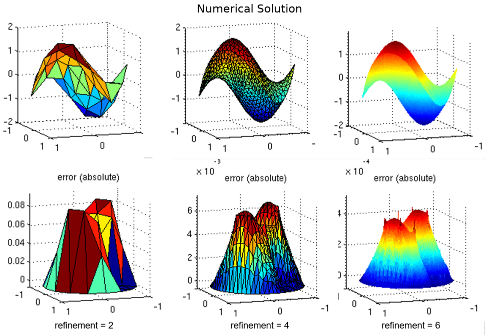

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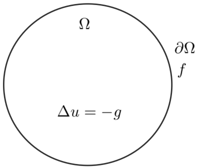

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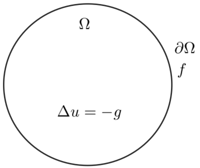

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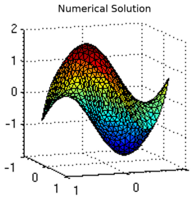

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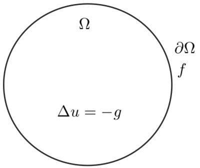

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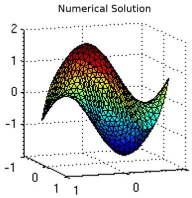

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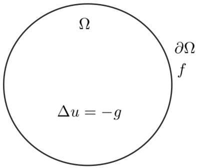

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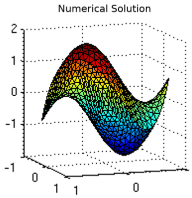

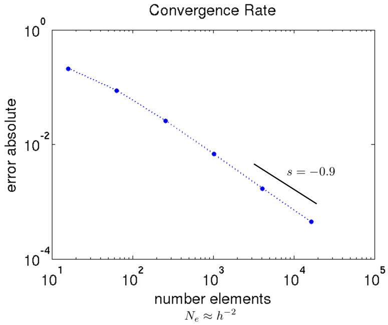

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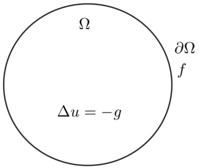

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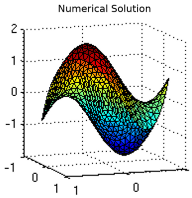
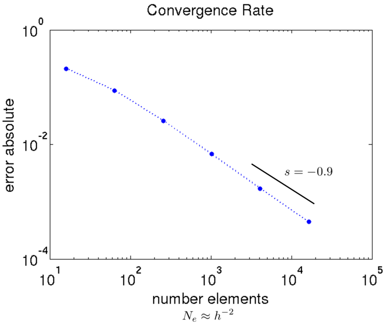
Log-log plots yield information on convergence rate





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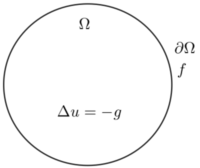

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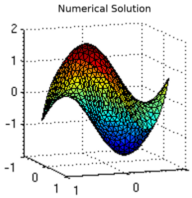
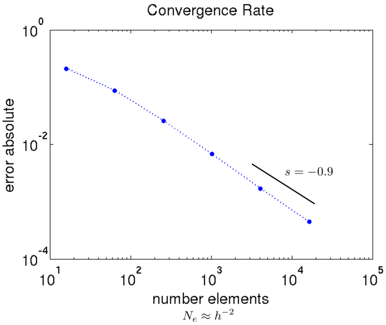
Log-log plots yield information on convergence rate $\epsilon = Ch^r$





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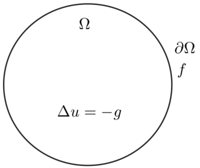

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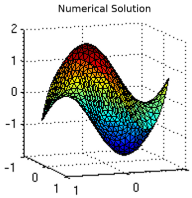
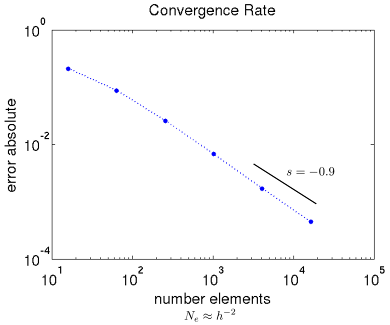
Log-log plots yield information on convergence rate $\epsilon = Ch^r \to \log(\epsilon) = \log(h)r + \log(C)$





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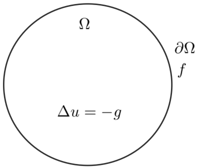

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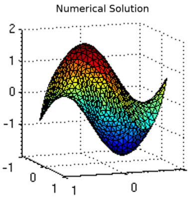
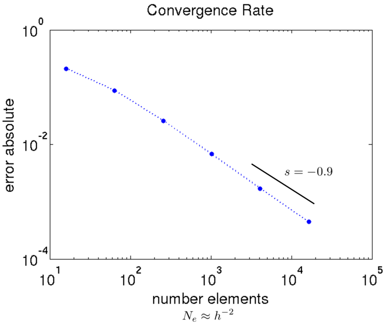
Log-log plots yield information on convergence rate

$\epsilon = Ch^r \rightarrow \log(\epsilon) = \log(h)r + \log(C) \Rightarrow \ -r/2 = s \sim -0.9$





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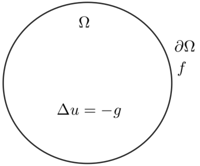

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Log-log plots yield information on convergence rate

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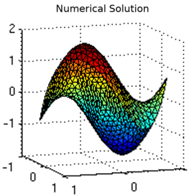

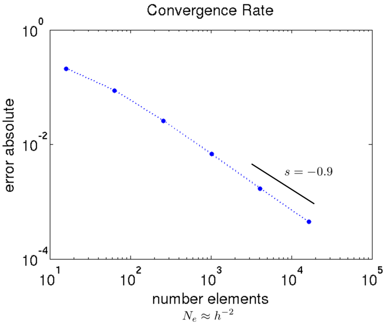

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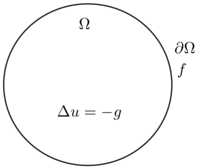

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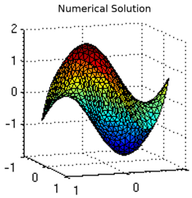
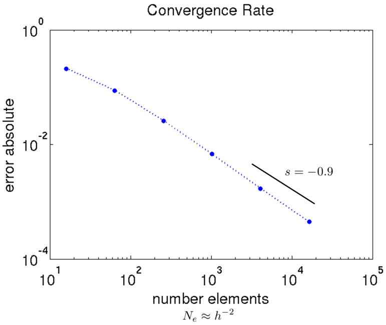
Log-log plots yield information on convergence rate

$\epsilon = Ch^r \rightarrow \log(\epsilon) = \log(h)r + \log(C) \Rightarrow -r/2 = s \sim -0.9 \rightarrow r \sim 1.8.$

Indicates 2^{nd}-order convergence rate.





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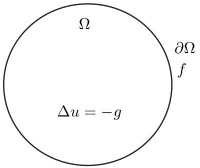

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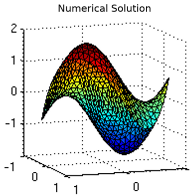

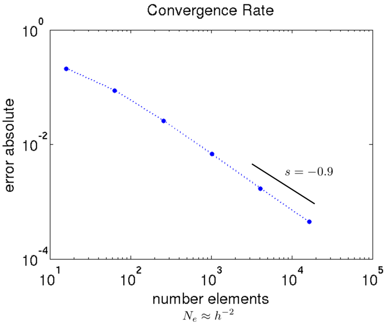

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Need to develop theory to predict from element properties.