Mixed Methods

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206D: Finite Element Methods
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When $\mathcal{L}$ contains only bilinear and quadratic expressions in u and λ, we obtain a saddle point problem.

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We need to establish conditions for this to be an isomorphism.

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(i) $\inf_{\mu \in M} \sup_{v \in X} \frac{b(v,\mu)}{\|v\| \|\mu\|} \geq \beta > 0$.
(ii) The operator $B : V^{\perp} \to M'$ is an isomorphism and $\|Bv\| \geq \beta \|v\|, \ \forall v \in V^{\perp}$.

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Proof:
The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma.
We show (iii) $\Rightarrow$ (ii). For $v \in V^{\perp}$ let $g \in V^0$ defined by mapping $w \mapsto (v, w)$.

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The equivalence of (i) and (iii) follows from the considerations in the previous Inf-Sup Lemma.
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Saddle Point Problems

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Saddle Point Problems

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Referred to as the Brezzi Conditions or Ladyzhenskaya-Babuska-Brezzi (LBB-Conditions).

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Mixed Finite Element Methods

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Remark: Also referred to as the Inf-Sup Conditions.

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Poisson Problem: Saddle-Point Formulation

Poisson Problem: Mixed Methods

Poisson Problem: Mixed Formulation

Find $(\sigma, u) \in L_2(\Omega)^d \times H_0^1(\Omega)$ so that

$$\begin{aligned} (\sigma, \tau)_{0,\Omega} - (\tau, \nabla u)_{0,\Omega} &= 0, \ \forall \tau \in L_2(\Omega)^d \\ -(\sigma, \nabla v)_{0,\Omega} &= -(f, v)_{0,\Omega}, \ \forall v \in H_0^1(\Omega). \end{aligned}$$

Poisson Problem: Saddle-Point Formulation

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Poisson Problem: Saddle-Point Formulation

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Saddle-Point Problem:

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Saddle-Point Problem:

$$a(\sigma, \tau) - b(\tau, v) = 0$$

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This establishes stability of the formulation.

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We can obtain stable Finite Element discretizations for triangulations $\mathcal{T}_h$.

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Poisson Problem: Stable Mixed Finite Element Spaces

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$$X_h := \left(\mathcal{M}^{k-1}\right)^d = \{\sigma_h \in L_2(\Omega)^d; \sigma_h|_T \in \mathcal{P}_{k-1}, \; \forall T \in \mathcal{T}_h\}$$

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Poisson Problem: Mixed Methods

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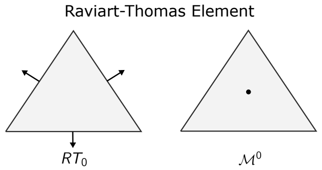
Poisson Problem: Mixed Methods

Raviart-Thomas Element

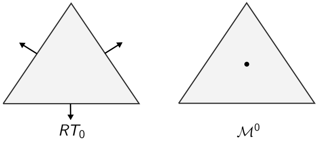


Poisson Problem: Mixed Methods

Raviart-Thomas Element

$$X_h \quad := \quad RT_k := \left\{ \tau \in L_2(\Omega)^2;\ \tau|_T = \begin{pmatrix} a_T \\ b_T \end{pmatrix} + c_T \begin{pmatrix} x \\ y \end{pmatrix},\ a_T, b_T, c_T \in \mathcal{P}_k,\ \forall T \in \mathcal{T}_h, \tau \cdot n \in \tilde{C}(\partial T) \right\}$$

Raviart-Thomas Element



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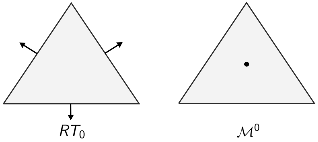

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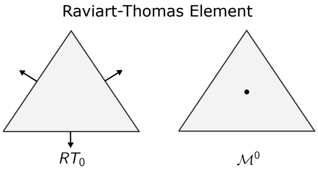

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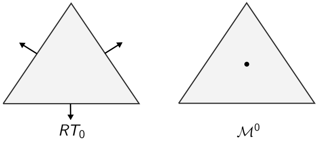
These can be shown to satisfy the Inf-Sup Condition for the Poisson Problem Mixed Formulation.

Raviart-Thomas Element



Poisson Problem: Mixed Methods

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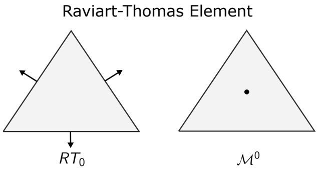

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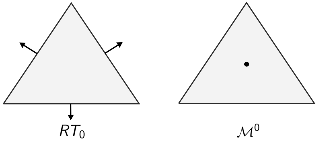
$$p(x, y) = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} x \\ y \end{pmatrix}.$$

The $n \cdot p$ is constant on $\alpha x + \beta y = c_0$ when n orthogonal to the line.

Raviart-Thomas Element



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Edge values determine the polynomial p.

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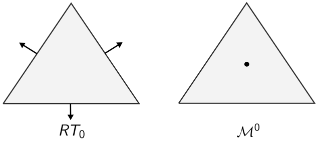

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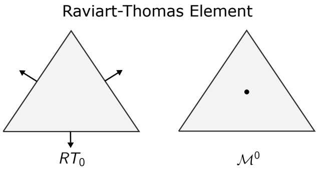

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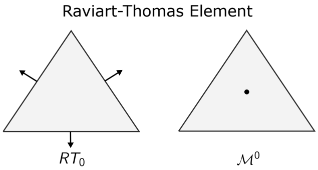
The $n \cdot p$ is constant on $\alpha x + \beta y = c_0$ when n orthogonal to the line. Edge values determine the polynomial p. Formally, elements are triple

$$\left(T, (\mathcal{P}_0)^2 + \mathbf{x} \cdot \mathcal{P}_0,\ n_i \cdot p(z_i), i = 1, 2, 3,\ z_i \text{ is edge midpoint.}\right)$$



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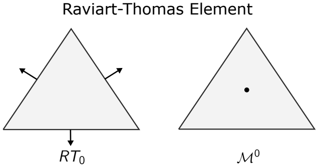

Poisson Problem: Raviart-Thomas Element

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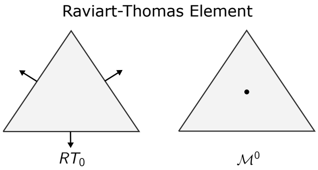

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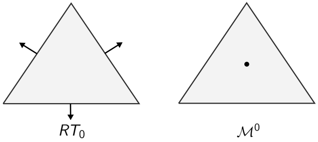

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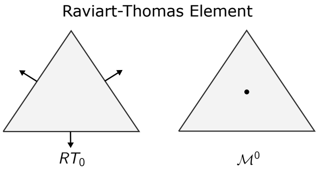

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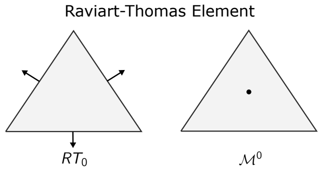

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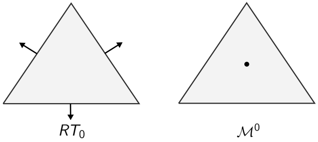

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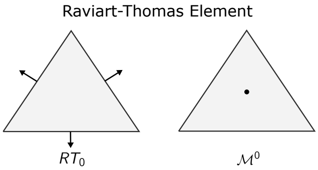

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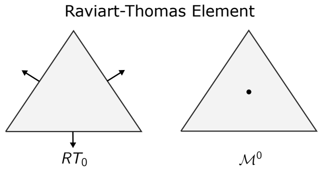

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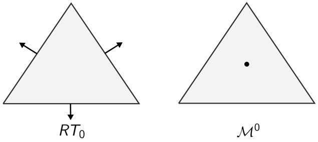

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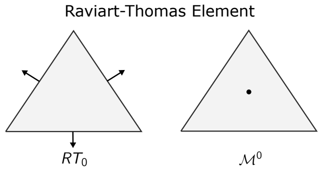

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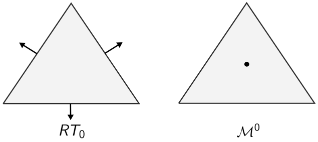

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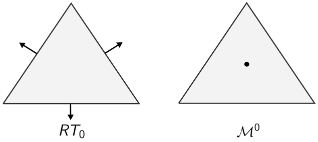

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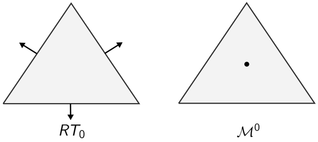
Inf-Sup Condition must be established.

Raviart-Thomas Element



Poisson Problem: Raviart-Thomas Element

Lemma: The Inf-Sup Condition

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By taking $|v|_{1,h}^2 = |v|_{1,h} c^{-1/2} \|\tau\|_{0,h}$, we have $b(\tau, v) \geq c^{-1/2} |v|_{1,h} \|\tau\|_{0,h}$.

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Poisson Problem: Raviart-Thomas Element

Lemma: The Inf-Sup Condition

The bilinear form b with the RT-elements satisfies

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This gives the Brezzi Condition for b.

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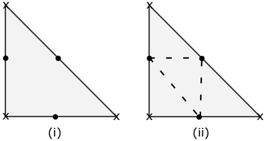
Stokes Hydrodynamic Equations: Taylor-Hood Element

Consider triangulation $\mathcal{T}_h$ and polymomial shape spaces $\mathcal{P}_j$.

Taylor-Hood Element



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Taylor-Hood Elements: Stability achieved by velocity field in polynomial space larger degree than the pressure space.

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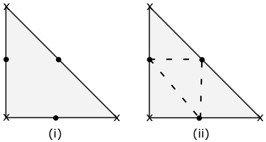

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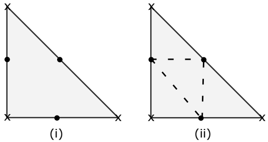

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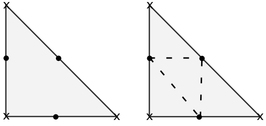

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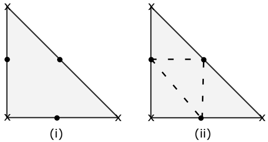

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Modified Taylor-Hood Element: Use piece-wise linear functions on sub-triangles (macro element)

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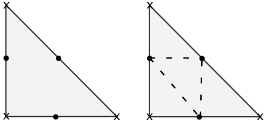

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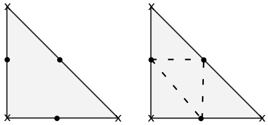

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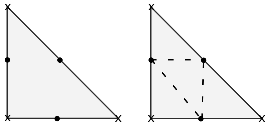

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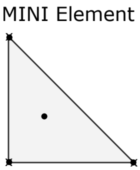
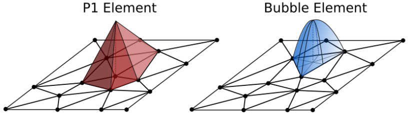
Figure: x denotes pressure values, · denotes velocity values.

Stokes Hydrodynamic Equations: MINI Element

MINI Elements: Achieves stability by using interior "bubble" elements.





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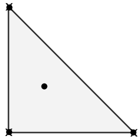

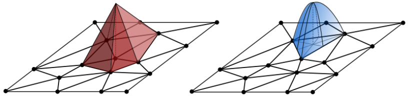

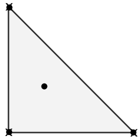

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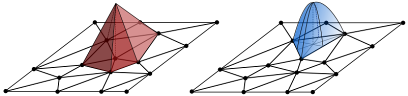

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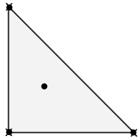

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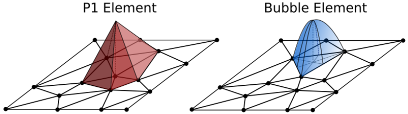

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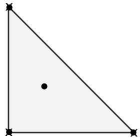

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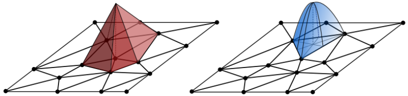

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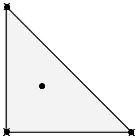

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