Ritz-Galerkin Approximation

Paul J. Atzberger

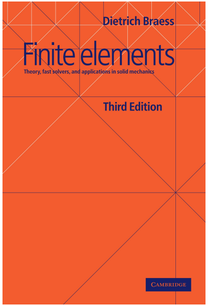
206D: Finite Element Methods
University of California Santa Barbara

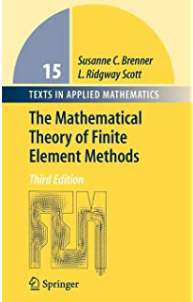
MATH 206D: Finite Element Methods

Welcome to MATH 206D: Finite Element Methods!

We will use the following books:

- *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics (third edition)*, D. Braess.
- *The Mathematical Theory of Finite Element Methods (third edition)*, S. Brenner and R. Scott.





For more information, see the course website:

http://teaching.atzberger.org/

I look forward to working with you this quarter.

Introduction to Finite Element Methods

Variational Approach

Variational Principle

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Suggests "natural boundary conditions" $\rightarrow u'(0) = u'(1) = 0.$

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Implies PDE holds (strong form)

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We have "stiffness matrix" $[K]_{ij} = a(\phi_i, \phi_j)$ and "load vector" $[\mathrm{f}]_i = (-f, \phi_i)$.

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Still, need theory to show $u_S \to u$ as $\mathcal{S} \to \mathcal{V}$ (i.e. we recover solution to the PDE in limit).

Linear Elements

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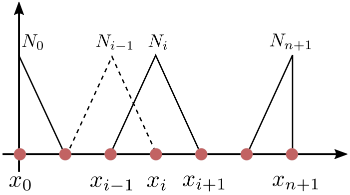
Consider space $\mathcal{S}$ generated by

$$v(x) = \sum_{i=1}^{n+1} v_i \phi_i(x)$$



Linear Elements

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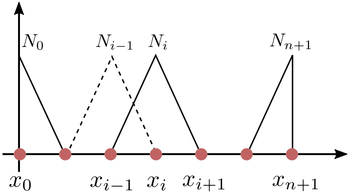

where $\phi_i(x) = N_i(x)$,

$$N_i(x) = \left\{ \begin{array}{ll} (x - x_{i-1})/h_{i-1}, & x \in [x_{i-1}, x_i] \\ (x_{i+1} - x)/h_i, & x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{array} \right\}$$

(Hat Functions).

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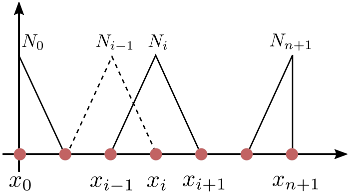

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Here, $h_i = x_{i+1} - x_i$ and $N_i(x_j) = \delta_{ij}$.

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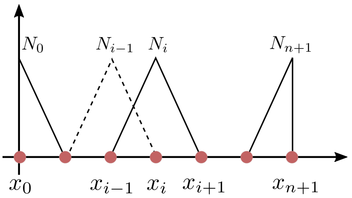

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Here, $h_i = x_{i+1} - x_i$ and $N_i(x_j) = \delta_{ij}$.

Mesh: $x_0, x_1, \ldots, x_{n+1}$. **Elements:** $e_i = \{x | x_{i-1} \leq x \leq x_{i+1}\}$. **Shape Functions:** $N_i(x)$.

Linear Elements

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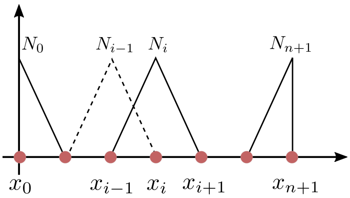

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Let $\mathcal{S} = \{v | v \in C[0, L], v(x) = \sum_{i=1}^{n} v_i N_i(x)\}$, referred to as the **shape space**.

Linear Elements

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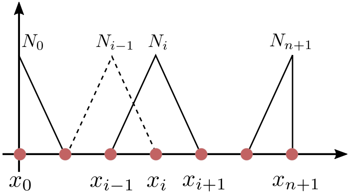

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We would like to carry-out the Ritz-Galerkin approximations over this space.

Linear Elements

Shape functions:

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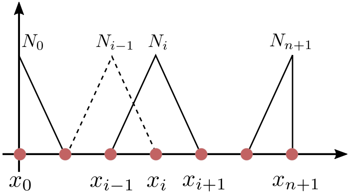

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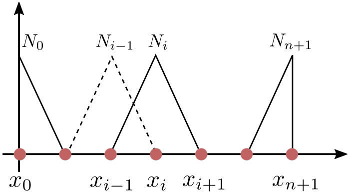

Consider the heat equation in 1D on $[0, L]$

$$\left\{ \begin{array}{ll} \frac{d^2u}{dx^2} = -f(x), & x \in [0, L] \\ u(0) = T_1, u(L) = T_2, & x \text{ on boundary} \end{array} \right.$$

Linear Elements

Shape functions:

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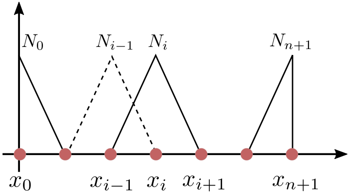

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Boundary conditions: $u_0 = T_1$ and $u_{n+1} = T_2$ throughout.

Linear Elements

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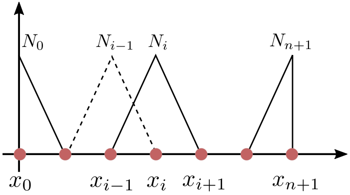

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Boundary conditions: $u_0 = T_1$ and $u_{n+1} = T_2$ throughout. Weak form

$$a(u, v) = (-f, v), \ \forall v \in \mathcal{V}, \qquad \text{WLOG } \mathcal{V} = \{v | v \in C[0, L], v(0) = v(L) = 0\}$$

Linear Elements

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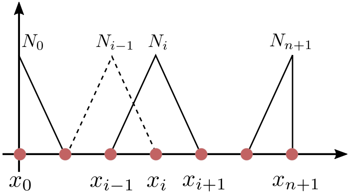

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We obtain Ritz-Galerkin Approximation by considering finite dimensional problem

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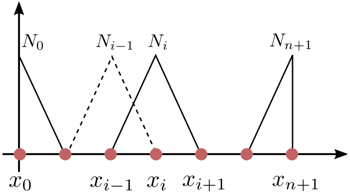

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To obtain stiffness matrix K and load vector f, we need to compute the inner-products.

Linear Elements

Shape functions:

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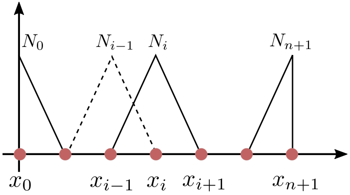

Ritz-Galerkin Approximation

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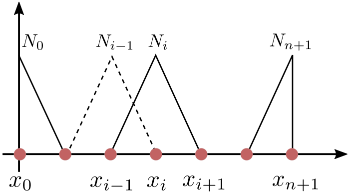

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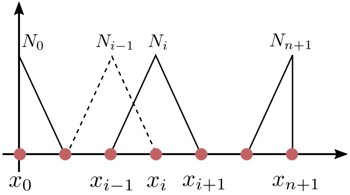

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Linear Elements

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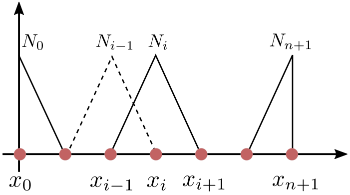

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Linear Elements

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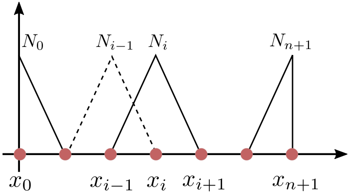

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Linear Elements

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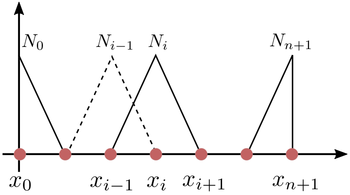

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$$(-f, N_i) = \int_{x_{i-1}}^{x_{i+1}} f(x) N_i(x) dx.$$

When $f = \sum_{i=0}^{n+1} f_i N_i(x)$, compute via "mass matrix" $M_{ij} = (N_i, N_j)$, and $[\mathrm{f}]_i = -M_{ij} f_j$.

Linear Elements

Shape functions:

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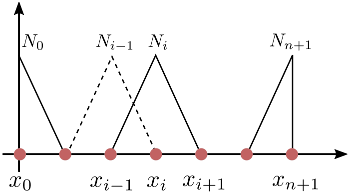

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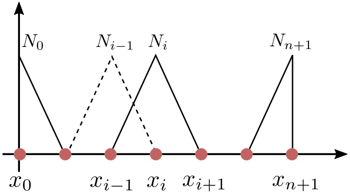

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Stiffness matrix when $h_i = h_0 = h$ and load vector when $f(x) = f_0$,

$$K = \frac{1}{h} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & -2 \end{bmatrix},$$

Linear Elements

Shape functions:

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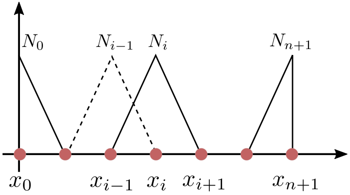

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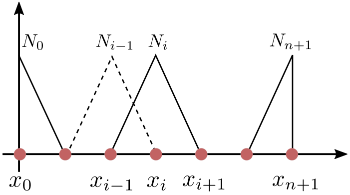
Linear Elements

Shape functions:

$$N_i(x) = \left\{ \begin{array}{ll} (x - x_{i-1})/h_{i-1}, & x \in [x_{i-1}, x_i] \\ (x_{i+1} - x)/h_i, & x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{array} \right\}$$



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Stiffness matrix when $h_i = h_0 = h$ and load vector when $f(x) = f_0$,

$$K = \frac{1}{h} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & -2 \end{bmatrix}, \mathrm{f} = -h \begin{bmatrix} f_0 \\ f_0 \\ f_0 \\ \vdots \\ f_0 \end{bmatrix}$$

The $Ku = \mathrm{f}$ equivalent system to Finite Difference Method for heat equation.

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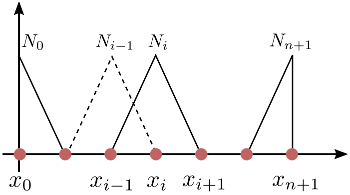

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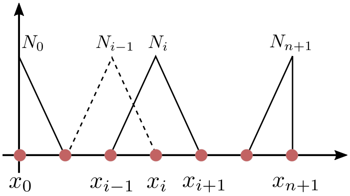

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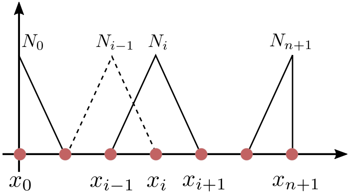

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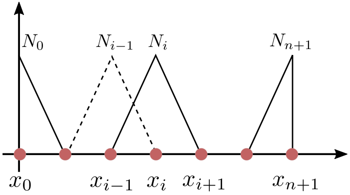

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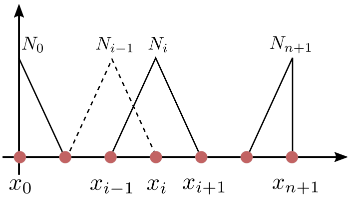

Example (linear elements):
The Green's function for $-d^2u/dx^2 = f$ is given by

$$G(x, x_0) = \left\{ \begin{array}{ll} x, & x < x_0 \\ x_0, & x_0, \text{ otherwise} \end{array} \right\}, \quad \frac{dG}{dx} = \left\{ \begin{array}{ll} 1, & x < x_0 \\ 0, & x_0, \text{ otherwise} \end{array} \right\}, \quad \frac{d^2G}{dx^2} = -\delta(x - x_0).$$

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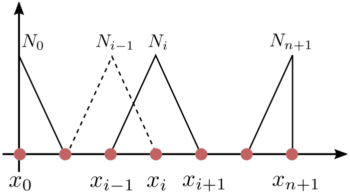

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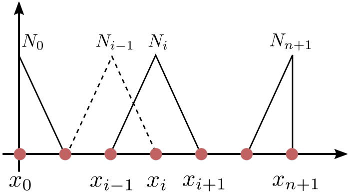
The solution u above can be expressed as

$$u(x) = \int G(x, y) f(y) dy.$$

Error Estimates

Example (linear elements) (continued)



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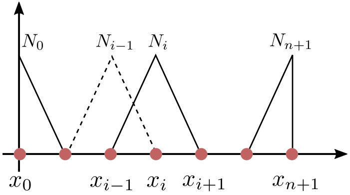

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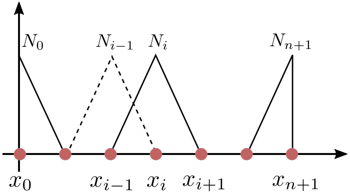

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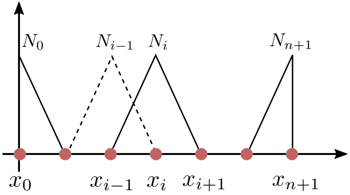

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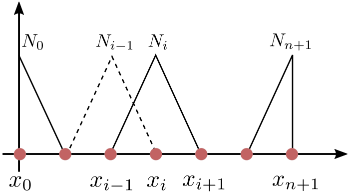

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This means u_S is piece-wise linear with $u_S(x_i) = u(x_i)$. We denote $u_S = u_I$ where u_I is the linear interpolation of the solution.

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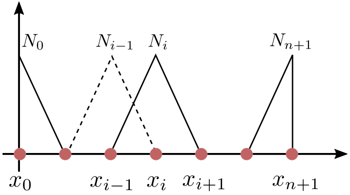
Lemma: The error of linear interpolation satisfies
$\|u - u_I\|_\infty \leq Ch^2 \|u''\|_\infty$.



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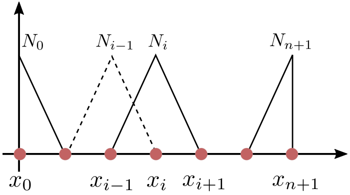

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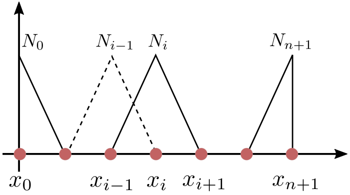

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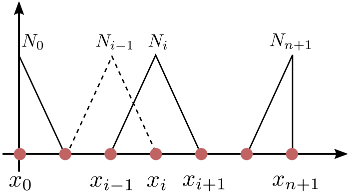

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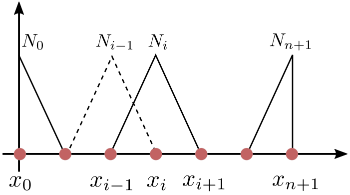

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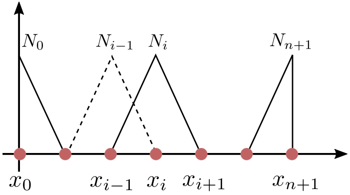

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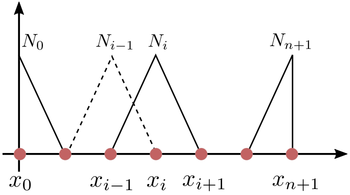

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Key is to design function spaces and study their interpolation theory, since this indicates FEM errors.